Math 320 Midterm 1 Practice Problems

Instructions

(i) Midterm solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

(iii) Self-contained solutions are optimal. If in doubt whether to include the proof of a statement, ask your instructor.

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1. Let X and Y be sets and let $f\colon X \to Y$. Suppose that X is uncountable and that for all $y \in Y$, the set $\{x \in X \colon f(x) = y\}$ is countable. Prove that Y is uncountable.

2. Let $f : [0,1] \to [0,1]$ be a non-decreasing function, i.e. $f(x) \leq f(y)$ whenever $x \leq y$. Let $D \subset [0,1]$ be the set of points where f is discontinuous. Prove that D is countable.

3. A number $\alpha \in \mathbb{R}$ is called *algebraic* if there exists a non-zero polynomial $P(x) = a_n x^n + \ldots + a_0$ with integer coefficients so that $P(\alpha) = 0$. A number $\alpha \in \mathbb{R}$ is called *transcendental* if it is non-algebraic. Prove that there exists at least one transcendental number.

4. Let $X = \mathbb{N} \cup \{a\}$, where a is an element not contained in $\mathbb{N}$. We will consider the metric space (X, d), where d is defined as follows: $d(a,a) = 0$; if $n \in \mathbb{N}$, then
$$d(a,n) = d(n,a) = 2^{-n+1};$$
if $n, m \in \mathbb{N}$ then
$$d(n,m) = \sum_{j=\min(n,m)}^{\max(n,m)} 2^{-j}.$$
You do not have to prove that d is a metric. Let $E = X$. What is the set E' of limit points of E? Prove that your answer is correct.

5. Let $f\colon \mathbb{N} \to \mathbb{R}$ with $\sum_{i=1}^{\infty} |f(i)| = A$ and $\sum_{i=1}^{\infty} |f(i)|^2 = 1$. Define $\operatorname{supp}(f) = \{n \in \mathbb{N} \colon f(n) \neq 0\}$. Prove that
$$|\operatorname{supp}(f)| \geq A^2.$$
Here $|\operatorname{supp}(f)|$ denotes the cardinality of the set $\operatorname{supp}(f)$.

6. Show that any collection of pairwise disjoint, nonempty open intervals in $\mathbb{R}$ is at most countable.

7. Recall the construction of the "Cantor middle-third set" $\mathcal{C}$ as given in Problem 3 of Homework 4. Determine whether the following statement is true or false, with adequate justification. "There exists an open interval I in the Cantor middle-third set $\mathcal{C}$."

8. We say that a subset A of a metric space (M,d) is *bounded* if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. The *diameter* of a set $A \subset M$ is given by
$$\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\}.$$
Show that A is bounded if and only if its diameter is finite.

9. Give an example where

$$\operatorname{diam}(A \cup B) > \operatorname{diam}(A) + \operatorname{diam}(B).$$

If $A \cap B \neq \emptyset$, then show that

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

10. Does there exist a metric ρ on $\mathbb{R}$ such that any convergent sequence in the usual metric remains so in $(\mathbb{R}, \rho)$, but the sequence $\{n : n \in \mathbb{N}\}$ is bounded in $(\mathbb{R}, \rho)$?

Solution key

Disclaimer

(i) Some of the following discussion is intended to provide pointers for the solutions only. Flesh out these ideas in greater detail to arrive at a complete solution.

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1. *Solution:* Prove the contrapositive. Suppose that Y is countable. Then $X = \bigcup_{y \in Y} \{x \in X \colon f(x) = y\}$ is a countable union of countable sets, and is thus countable.

2. *Hint:* Recall that if f is non-decreasing, then $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ always exist (though they need not be equal).

   Solution: Since f is non-decreasing, for every $a \in (0,1)$ we have that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist. Thus f is discontinuous at a if and only if $\lim_{x \to a^-} f(x) < \lim_{x \to a^+} f(x)$. Such discontinuities are called "jump discontinuities."

   For each $n \in \mathbb{N}$, let $X_n = \{a \in [0,1] \colon \lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x) > 1/n\}$. Then $D = \bigcup_{n=1}^{\infty} X_n$. If D is uncountable, then at least one X_n must be uncountable (since a countable union of countable sets is countable). In particular, at least one of these sets X_n must be infinite. But since $f(0) \geq 0$ and $f(1) \leq 1$, we must have that $|X_n| \leq n$ for each $n \in N$, and in particular, each set X_n must be finite.

3. *Solution:* First, observe that for each $n \in \mathbb{N}$, the set of polynomials of degree $\leq n$ with integer coefficients is countable, since it can be put in bijective correspondence with $\mathbb{Z}^{n+1}$ via the bijection $(a_0, ... a_n) \mapsto P(x) = a_n x^n + \ldots + a_0$. Thus the set of polynomials with integer coefficients is a countable union of countable sets, and is thus countable. For each polynomial P, let $S_P = \{x \in \mathbb{R} \colon P(x) = 0\}$. This set is finite (indeed, it has cardinality at most the degree of P). Thus $A = \bigcup_P S_P$ is a countable union of countable sets, and is thus countable, where the union is taken over all non-zero polynomials with integer coefficients. However, the set A is precisely the set of algebraic numbers. We conclude that A is countable. If $A = \mathbb{R}$ then this would imply that $\mathbb{R}$ is countable, which we know is not the case. Thus $\mathbb{R} \backslash A$ is non-empty, i.e. there exists at least one transcendental number.

4. *Solution:* We will prove that $E' = \{a\}$. Indeed, let $n \in \mathbb{N}$. Then selecting $r = 2^{-n-1}$, we see that $N_r(n) = \{n\}$, so n is not in E'. On the other hand, for every $r > 0$ there exists a natural number m so that $2^{-m} < r$, so $m \in N_r(a)$ and thus $N_r(a) \cap E$ contains a point other than a. We conclude that $E' = \{a\}$.

5. *Solution:* If $\operatorname{supp}(f)$ is infinite then the result is immediately true. If $\operatorname{supp}(f)$ is finite, then let $n = |\operatorname{supp}(f)|$. Without loss of generality, we can assume that $\operatorname{supp}(f) = \{1, \ldots, n\}$.

   By Cauchy-Schwarz, we have

$$A^2 = \Big(\sum_{i=1}^{n} f(i)\Big)^2 = \Big(\sum_{i=1}^{n} 1 f(i)\Big)^2 \leq \Big(\sum_{i=1}^{n} 1^2\Big)\Big(\sum_{i=1}^{n} f(i)^2\Big) = n\Big(\sum_{i=1}^{n} f(i)^2\Big) = n.$$

   Taking square roots of both sides, we obtain

$$A \leq \sqrt{n},$$

   as desired.

6. *Hint :* Each interval contains a rational!

7. *Solution:* False. The Cantor middle-third set C is *nowhere dense*, i.e. contains no nonempty open intervals. We will show that

$$\text{given any } x, y \in C,\ x < y, \text{ there exists } z \in [0,1] \setminus C \text{ such that } x < z < y. \tag{1}$$

Recall that

$$C = \bigcap_{n=1}^{\infty} C_n,$$

where C_n, the set obtained at the n-th step of the Cantor construction, is a disjoint union of 2^n closed intervals (called n-th stage *basic intervals*), each of length 3^{-n}. Given x, y as above, there exists a largest positive integer n such that both x and y lie inside a common n-th stage basic interval, say $I = [a, b]$. At the $(n+1)$-th step, I is decomposed into three equal and disjoint pieces

$$I = \bigcup_{j=1}^{3} I_j, \quad \text{with } I_1 = \left[a, a + \frac{b-a}{3}\right], \quad I_3 = \left[b - \frac{b-a}{3}, b\right]$$

and the middle third portion I_2 is thrown away. In particular, $z = a + (b-a)/2 = (a+b)/2 \notin C$. By the maximality of n, we also know that $x \in I_1$ and $y \in I_3$, proving (1).

8. *Hint:* Use the triangle inequality.

9. *Solution:* Let $A = [0,1]$, $B = [99, 100]$. Then

$$\operatorname{diam}(A \cup B) = 100 > 1 + 1 = \operatorname{diam}(A) + \operatorname{diam}(B).$$

Suppose that $x_0 \in A \cap B$. Then for any $x, y \in A \cup B$ such that $x \in A$ and $y \in B$, we obtain from the triangle inequality that

$$d(x,y) \le d(x, x_0) + d(y, x_0) \le \operatorname{diam}(A) + \operatorname{diam}(B).$$

If x and y both lie in A (or in B), the inequality is trivially true because $d(x,y) \le \operatorname{diam}(A) \le \operatorname{diam}(A) + \operatorname{diam}(B)$. Thus, $\operatorname{diam}(A) + \operatorname{diam}(B)$ is an upper bound for the set $\{d(x,y) : x, y \in A \cup B\}$. Since $\operatorname{diam}(A \cup B) = \sup\{d(x,y) : x, y \in A \cup B\}$, the result follows.

10. *Hint:* Yes. Try $\rho = d/(1+d)$ (or a variant), where d is the usual metric.