# Math 421/510, Spring 2007, Final
# (Due date: Friday April 20)

## Instructions

- The final is due by 10:00 am on Friday April 20, and should be left in the instructor's departmental mailbox. **There will be no extensions for the final.**
- **All pending homework assignments should be turned in along with the final. No submissions will be accepted after 10:00 am on Friday April 20.**
- Unlike homework assignments, you must work on the final on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof. Results quoted in class whose verification was left as exercises should be proved in detail.

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1. (i) We know that weak convergence of a sequence does not imply strong convergence. Show however that the following variant is true. Suppose that $\{x_n\}$ is a sequence in a normed space $\mathfrak{X}$ and $x \in \mathfrak{X}$ such that $\varphi(x_n) \to \varphi(x)$ for all $\varphi \in \mathfrak{X}^*$. Show that for each $\epsilon > 0$ and $m \geq 1$, there exists a convex combination $y = \sum_{n \geq m} \lambda_n x_n$ (i.e., $\lambda_n \geq 0$ and $\sum_{n \geq m} \lambda_n = 1$) such that $||x - y|| < \epsilon$.

   (ii) Use the result in part (i) to derive the following statement. Let $\{f_n\}$ be a bounded sequence in $C(X)$, such that $f_n(x) \to f(x)$ for every $x \in X$, where $f \in C(X)$. Then for each $\epsilon > 0$ there is a convex combination $g = \sum_n \lambda_n f_n$ such that $||f - g||_\infty < \epsilon$.
2. Let $B$ denote the unit ball in $M[0,1]$, the space of all complex-valued regular Borel measures on $[0,1]$. For $\mu, \nu \in M[0,1]$, define
$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n} \left| \int_0^1 x^n d\mu - \int_0^1 x^n d\nu \right|.$$
   Show that $d$ is a metric on $M[0,1]$ that describes the weak$^*$ topology on $B$ but not on $M[0,1]$.
3. If $\mathfrak{X}$ is a Banach space, show that there is a compact space $X$ such that $\mathfrak{X}$ is isometrically isomorphic to a closed subspace of $C(X)$.