Math 421/510, Spring 2007, Midterm
(Due date: Friday March 2)

Instructions

- Every problem in the midterm is worth 25 points.
- The midterm will be collected at the end of lecture on Friday. **There will be no extensions for the midterm.**
- Unlike homework assignments, you must work on the midterm on your own. If you need hints or clarifications, please feel free to talk to the instructor.
- Answers should be clear, legible, and in complete English sentences. Solutions must be self-contained – only results proved in class can be used without proof.

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1. Show that a Banach space X is separable if its dual space X^* is separable. Is the converse true?
2. (a) Consider the vectors $(1, 2, 0, 0, \cdots)$ and $(1, 1, 1, 0, \cdots)$ in ℓ^2. Change the inner product on ℓ^2 so that it remains a Hilbert space and these two vectors are orthogonal.
   (b) In general given linearly independent vectors $\varphi_1, \cdots, \varphi_n$ in a Hilbert space $\mathcal{H}$, change the inner product on $\mathcal{H}$ so that it remains a Hilbert space and $\varphi_1, \cdots, \varphi_n$ become orthogonal.
3. Prove or disprove the following statement: Let K be a topological space that is compact and Hausdorff. If T is a bounded linear operator from $C(K)$ to ℓ^1, then T is compact.
4. This problem involves a quantitative notion related to extensions of linear operators. We say that a Banach space X is *injective* if for every Banach space Y and every subspace Z of Y and every continuous linear operator $T : Z \to X$, there exists an extension $\widetilde{T} : Y \to X$. The *extension constant* $e(X)$ is defined as

$$e(X) = \inf \left\{ c \;\middle|\; \begin{array}{l} \text{For every } Y \supset Z \text{ and } T : Z \to X, \text{ there exists} \\ \widetilde{T} : Y \to X \text{ such that } \widetilde{T}\big|_Z = T \text{ and } ||\widetilde{T}|| \leq c||T|| \end{array} \right\}.$$

   (a) Prove that $e(X) < \infty$ for any injective Banach space X.
   (b) Determine $e(\ell_\infty)$.