Math 421/510, Spring 2008
Homework Set 4
due on Monday February 25

Instructions

- You are encouraged to discuss homework problems among yourselves. Also feel free to ask the instructor for hints and clarifications. However the written solutions that you submit should be entirely your own.
- Answers should be clear, legible, and in complete English sentences. If you need to use results other than the ones discussed in class, state the result clearly with either a reference or a self-contained proof.

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1. Find an example of a linear functional on a normed vector space that is not continuous. Find an example of a discontinuous linear operator between two normed spaces X and Y, where $\dim(Y) > 1$.
2. Recall the multiplication operator we discussed in class. We will study this operator in a bit more detail. Let (X, Ω, μ) be a σ-finite measure space. For any Ω-measurable function ϕ, define
$$M_\phi(f) = \phi f.$$
   (a) If $\phi \in L^\infty(\mu)$, we have seen that $M_\phi \in \mathcal{B}(L^p(\mu))$, with $||M_\phi|| \le ||\phi||_\infty$. Prove that in fact $||M_\phi|| = ||\phi||_\infty$.
   (b) Conversely, if for some $p \in [1, \infty]$,
$$\phi f \in L^p(\mu) \text{ whenever } f \in L^p(\mu),$$
   then show that $\phi \in L^\infty(\mu)$.
3. Let us consider another example in our list of continuous linear operators. Namely, if X and Y are compact Hausdorff spaces and $\tau : Y \to X$ a continuous map, let $A \in \mathcal{B}(C(X), C(Y))$ denote the operator that is "composition with τ".
   (a) Show that $||A|| = 1$.
   (b) Give a necessary and sufficient condition on τ so that A is injective.
   (c) Give such a condition for A to be surjective.
   (d) Give such a condition for A to be an isometry.
4. Let X be a normal locally compact space and F a closed subset of X. If $M = \{f \in C_0(X) : f\big|_F \equiv 0\}$, then $C_0(X)/M$ is isometrically isomorphic to $C_0(F)$.

5. Let $T : X \to U$ and $S : U \to W$ be linear operators on normed spaces. If X, U and W are vector spaces of finite and equal dimension, and ST is invertible, prove that S and T are both invertible. Give an example to show that the result is false if at least one of the vector spaces is infinite-dimensional.
6. We used the Baire Category Theorem in class to deduce that any Hamel basis of an infinite-dimensional vector space must be uncountable. We will sketch a proof of the theorem in this exercise.

   Theorem 1 (Baire Category Theorem)**.** *A complete metric space is of the second category. That is, if M is a complete metric space and if we write $M = \bigcup_{n=1}^{\infty} E_n$, then the closure of some E_n contains an open ball. Equivalently, if $\{G_n\}$ is a sequence of dense, open sets in M, then $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$; in fact, $\bigcap_{n=1}^{\infty} G_n$ is dense in M.*

   (a) Prove that any complete metric space satisfies the "nested set property", namely if $F_1 \supset F_2 \supset F_3 \supset \cdots$ is a decreasing sequence of nonempty closed sets in M with $\operatorname{diam}(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, and consists exactly of one point. Incidentally, the nested set property is equivalent to completeness, though you will probably not need the converse statement for this problem.
   (b) Prove the Baire Category Theorem. (Hint: Start with a collection $\{G_n\}$ of dense open sets in M. Fix any $x_0 \in M$ and any open ball B_0 containing x_0. You should prove that $\bigcap_{n=1}^{\infty} G_n \cap B_0 \neq \emptyset$ (why?). Using the properties of G_1, find an open ball B_1 of diameter no more than 1 such that $\bar{B}_1 \subset B_0 \cap G_1$. Repeat the same argument to find an open ball B_2 with $\operatorname{diam}(B_2) \leq \frac{1}{2}$ such that $\bar{B}_2 \subset B_1 \cap G_2$. Iterate the process, and apply the nested set property on the $\bar{B}_j$-s.)
7. We stated in class that $C[0,1]$ has a Schauder basis. The goal of this exercise is to find one. Enumerate the dyadic rationals (i.e. rationals of the form $\frac{k}{2^m}$, where k, m are non-negative integers, and k is odd) in the usual way, as follows :

$$t_0 = 0,\ t_1 = 1,\ t_2 = \frac{1}{2},\ t_3 = \frac{1}{4},\ t_4 = \frac{3}{4}, \cdots.$$

   Set $f_0 \equiv 1$, $f_1(x) = x$. If $t_n = k_n/2^{m_n}$ for $m_n \geq 1$, and $\gcd(k_n, 2) = 1$, define the function $f_n : [0,1] \to \mathbb{R}$ as the one that satisfies $f_n(0) = f_n(1) = 0$, $f_n(t_n) = 1$, $f_n(t_n - 2^{-m_n}) = f_n(t_n + 2^{-m_n}) = 0$, and f_n interpolates linearly in between.
   (a) Show that the functions f_n are linearly independent.

(b) Show that span$\{f_0, \cdots, f_{2^m}\}$ is the space of all continuous, piecewise linear functions with "nodes" at the dyadic rationals $r2^{-m}$, $r = 0, 1, \cdots, 2^m$.

(c) Show that the functions $\{f_n\}$ have dense linear span in $C[0, 1]$.

(d) Show that $\{f_n\}$ is a Schauder basis for $C[0, 1]$.

(e) Given $f \in C[0, 1]$, give a geometric description of its Schauder approximations $\sum_{k=1}^{n} a_k f_k$.

Remark : We proved in class that the collection of monomials is not a Schauder basis for $C[0, 1]$. However, $C[0, 1]$ does admit a basis consisting entirely of polynomials (do you find this surprising?). We may come back to this later in the semester.

8. Show that the class of continuous functions on $[0, 1]$ is dense in $L^2[0, 1]$. Can you replace "continuous" by "infinitely smooth"? How about $[0, 1]$ by $\mathbb{R}^n$?