27. The area and moments of the region are

$$A = \int_0^\infty \frac{dx}{(1+x)^3} = \lim_{R\to\infty} \frac{-1}{2(1+x)^2}\Bigg|_0^R = \frac{1}{2}$$

$$M_{x=0} = \int_0^\infty \frac{x\,dx}{(1+x)^3} \qquad \text{Let } u = x+1,\quad du = dx$$

$$= \int_1^\infty \frac{u-1}{u^3}\,du$$

$$= \lim_{R\to\infty}\left(-\frac{1}{u} + \frac{1}{2u^2}\right)\Bigg|_1^R = 1 - \frac{1}{2} = \frac{1}{2}$$

$$M_{y=0} = \frac{1}{2}\int_0^\infty \frac{dx}{(1+x)^6} = \lim_{R\to\infty} \frac{-1}{10(1+x)^5}\Bigg|_0^R = \frac{1}{10}.$$

The centroid is $\left(1, \frac{1}{5}\right)$.

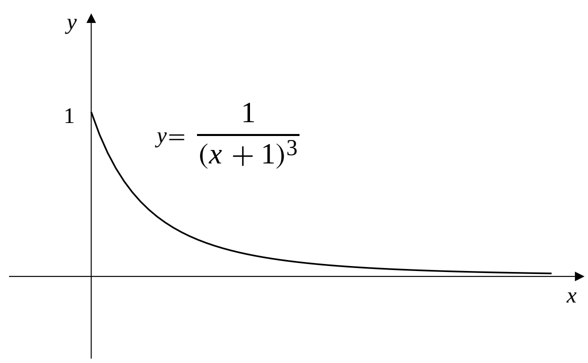

Fig. 5-27

21. By symmetry the centroid is $(1, -2)$.

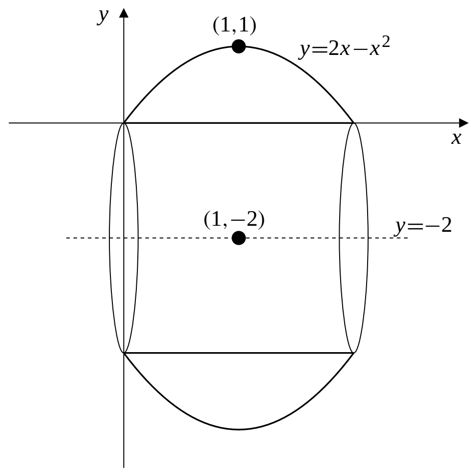

Fig. 5-21

4. The height of each triangular face is $2\sqrt{3}$ m and the height of the pyramid is $2\sqrt{2}$ m. Let the angle between the triangular face and the base be θ, then $\sin\theta = \sqrt{\dfrac{2}{3}}$ and $\cos\theta = \dfrac{1}{\sqrt{3}}$.

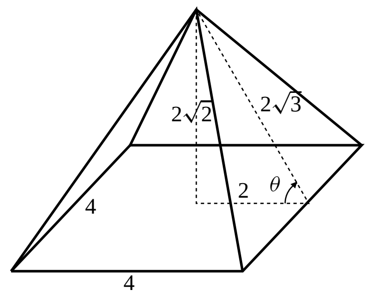

Fig. 6-4

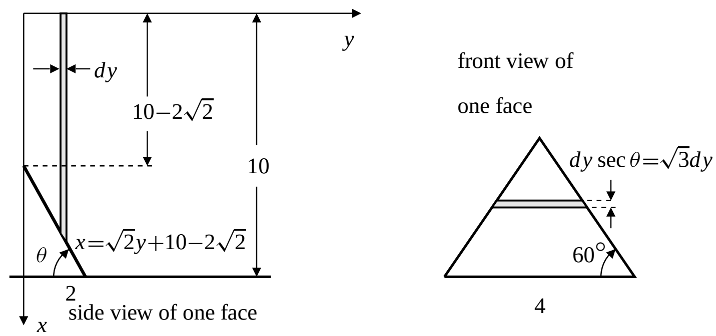

Fig. 6-4

A vertical slice of water with thickness dy at a distance y from the vertex of the pyramid exerts a force on the shaded strip shown in the front view, which has area $2\sqrt{3}y\,dy$ m^2 and which is at depth $\sqrt{2}y + 10 - 2\sqrt{2}$ m. Hence, the force exerted on the triangular face is

$$\begin{aligned} F &= \rho g \int_0^2 (\sqrt{2}y + 10 - 2\sqrt{2})2\sqrt{3}y\,dy \\ &= 2\sqrt{3}(9800)\left[\frac{\sqrt{2}}{3}y^3 + (5-\sqrt{2})y^2\right]\Bigg|_0^2 \\ &\approx 6.1495 \times 10^5 \text{ N}. \end{aligned}$$

6. The spring force is $F(x) = kx$, where x is the amount of compression. The work done to compress the spring 3 cm is

$$100 \text{ N}\cdot\text{cm} = W = \int_0^3 kx\,dx = \frac{1}{2}kx^2\Big|_0^3 = \frac{9}{2}k.$$

Hence, $k = \dfrac{200}{9}$ N/cm. The work necessary to compress the spring a further 1 cm is

$$W = \int_3^4 kx\,dx = \left(\frac{200}{9}\right)\frac{1}{2}x^2\Big|_3^4 = \frac{700}{9} \text{ N}\cdot\text{cm}.$$

12. Let the time required to raise the bucket to height h m be t minutes. Given that the velocity is 2 m/min, then $t = \frac{h}{2}$. The weight of the bucket at time t is $16\text{ kg} - (1\text{ kg/min})(t\text{ min}) = 16 - \frac{h}{2}$ kg. Therefore, the work done required to move the bucket to a height of 10 m is

$$\begin{aligned} W &= g\int_0^{10}\left(16 - \frac{h}{2}\right)dh \\ &= 9.8\left(16h - \frac{h^2}{4}\right)\Bigg|_0^{10} = 1323\text{ N}\cdot\text{m}. \end{aligned}$$

21. If X is distributed normally, with mean $\mu = 5,000$, and standard deviation $\sigma = 200$, then

$$\begin{aligned}
&\Pr(X \geq 5500) \\
&= \frac{1}{200\sqrt{2\pi}} \int_{5500}^{\infty} e^{-(x-5000)^2/(2\times 200^2)}\,dx \\
&\qquad\qquad \text{Let } z = \frac{x-5000}{200} \\
&\qquad\qquad dz = \frac{dx}{200} \\
&= \frac{1}{\sqrt{2\pi}} \int_{5/2}^{\infty} e^{-z^2/2}\,dz \\
&= \Pr(Z \geq 5/2) = \Pr(Z \leq -5/2) \approx 0.006
\end{aligned}$$

from the table in this section.

22. If X is the random variable giving the spinner's value, then $\Pr(X = 1/4) = 1/2$ and the density function for the other values of X is $f(x) = 1/2$. Thus the mean of X is

$$\mu = E(X) = \frac{1}{4}\Pr\left(X = \frac{1}{4}\right) + \int_0^1 x\, f(x)\, dx = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

Also,

$$E(X^2) = \frac{1}{16}\Pr\left(X = \frac{1}{4}\right) + \int_0^1 x^2\, f(x)\, dx = \frac{1}{32} + \frac{1}{6} = \frac{19}{96}$$

$$\sigma^2 = E(X^2) - \mu^2 = \frac{19}{96} - \frac{9}{64} = \frac{11}{192}.$$

Thus $\sigma = \sqrt{11/192}$.

31. The hyperbolas $xy = C$ satisfy the differential equation

$$y + x\frac{dy}{dx} = 0, \qquad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}.$$

Curves that intersect these hyperbolas at right angles must therefore satisfy $\frac{dy}{dx} = \frac{x}{y}$, or $x\,dx = y\,dy$, a separated equation with solutions $x^2 - y^2 = C$, which is also a family of rectangular hyperbolas. (Both families are degenerate at the origin for $C = 0$.)

24. $y(x) = 3 + \int_0^x e^{-y}\,dt \quad \Longrightarrow \quad y(0) = 3$

$\frac{dy}{dx} = e^{-y}, \qquad$ i.e. $e^y\,dy = dx$

$e^y = x + C \quad \Longrightarrow \quad y = \ln(x + C)$

$3 = y(0) = \ln C \quad \Longrightarrow \quad C = e^3$

$y = \ln(x + e^3).$

28. Given that $m\frac{dv}{dt} = mg - kv$, then

$$\begin{aligned}\int \frac{dv}{g - \frac{k}{m}v} &= \int dt \\ -\frac{m}{k}\ln\left|g - \frac{k}{m}v\right| &= t + C.\end{aligned}$$

Since $v(0) = 0$, therefore $C = -\frac{m}{k}\ln g$. Also, $g - \frac{k}{m}v$ remains positive for all $t > 0$, so

$$\begin{aligned}\frac{m}{k}\ln\frac{g}{g - \frac{k}{m}v} &= t \\ \frac{g - \frac{k}{m}v}{g} &= e^{-kt/m} \\ \Rightarrow \quad v = v(t) &= \frac{mg}{k}\left(1 - e^{-kt/m}\right).\end{aligned}$$

Note that $\lim_{t\to\infty} v(t) = \frac{mg}{k}$. This limiting velocity can be obtained directly from the differential equation by setting $\frac{dv}{dt} = 0$.