12. The ellipses $3x^2 + 4y^2 = C$ all satisfy the differential equation

$$6x + 8y\,\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x}{4y}.$$

A family of curves that intersect these ellipses at right angles must therefore have slopes given by $\dfrac{dy}{dx} = \dfrac{4y}{3x}$. Thus

$$\begin{aligned} 3\int \frac{dy}{y} &= 4\int \frac{dx}{x} \\ 3\ln|y| &= 4\ln|x| + \ln|C|. \end{aligned}$$

The family is given by $y^3 = Cx^4$.

8.

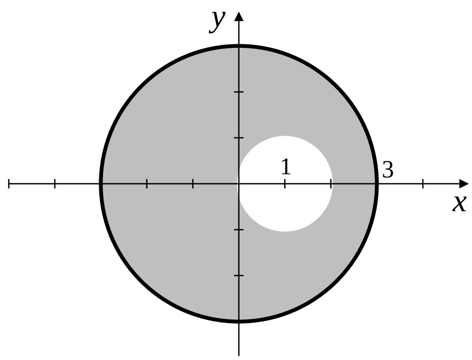

Fig. R-8

Let the disk have centre (and therefore centroid) at $(0, 0)$. Its area is 9π. Let the hole have centre (and therefore centroid) at $(1, 0)$. Its area is π. The remaining part has area 8π and centroid at $(\overline{x}, 0)$, where

$$(9\pi)(0) = (8\pi)\overline{x} + (\pi)(1).$$

Thus $\overline{x} = -1/8$. The centroid of the remaining part is $1/8$ ft from the centre of the disk on the side opposite the hole.

20. $y' + (\cos x)y = 2xe^{-\sin x}, \qquad y(\pi) = 0$

$$\mu = \int \cos x \, dx = \sin x$$

$$\frac{d}{dx}(e^{\sin x} y) = e^{\sin x}(y' + (\cos x)y) = 2x$$

$$e^{\sin x} y = \int 2x \, dx = x^2 + C$$

$$y(\pi) = 0 \;\Rightarrow\; 0 = \pi^2 + C \;\Rightarrow\; C = -\pi^2$$

$$y = (x^2 - \pi^2)e^{-\sin x}.$$

17.

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\text{mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/2\sigma^2}\, dx \qquad \text{Let z} = \frac{x-\mu}{\sigma}$$

$$dz = \frac{1}{\sigma}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-z^2/2}\, dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz = \mu$$

$$\text{variance} = E\big((x-\mu)^2\big)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-z^2/2}\, dz = \sigma \text{Var}(Z) = \sigma$$

9. A layer of water between depths y and $y + dy$ has volume $dV = \pi(a^2 - y^2)\,dy$ and weight $dF = 9,800\pi(a^2 - y^2)\,dy$ N. The work done to raise this water to height h m above the top of the bowl is

$$dW = (h + y)\,dF = 9,800\pi(h + y)(a^2 - y^2)\,dy \text{ N·m.}$$

Thus the total work done to pump all the water in the bowl to that height is

$$\begin{aligned} W &= 9,800\pi \int_0^a (ha^2 + a^2y - hy^2 - y^3)\,dy \\ &= 9,800\pi \left[ha^2y + \frac{a^2y^2}{2} - \frac{hy^3}{3} - \frac{y^4}{4}\right]\Bigg|_0^a \\ &= 9,800\pi \left[\frac{2a^3h}{3} + \frac{a^4}{4}\right] \\ &= 9,800\pi a^3 \frac{3a + 8h}{12} = 2450\pi a^3 \left(a + \frac{8h}{3}\right) \text{ N·m.} \end{aligned}$$

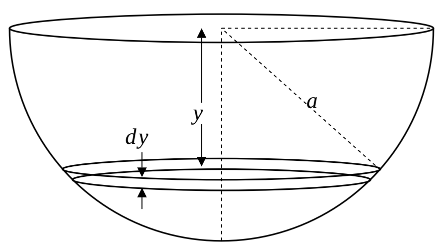

Fig. 6-9