6. The area of revolution of $y = \sqrt{x}$, $(0 \le x \le 6)$, about the x-axis is

$$\begin{aligned} S &= 2\pi \int_0^6 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_0^6 \sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx \\ &= 2\pi \int_0^6 \sqrt{x + \frac{1}{4}}\, dx \\ &= \frac{4\pi}{3}\left(x + \frac{1}{4}\right)^{3/2}\Bigg|_0^6 = \frac{4\pi}{3}\left[\frac{125}{8} - \frac{1}{8}\right] = \frac{62\pi}{3} \text{ sq. units.} \end{aligned}$$

15.

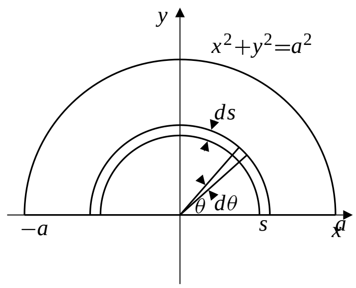

Fig. 4-15

Consider the area element which is the thin half-ring shown in the figure. We have

$$dm = ks\,\pi s\,ds = k\pi\,s^2\,ds.$$

Thus, $m = \dfrac{k\pi}{3}\,a^3$.
Regard this area element as itself composed of smaller elements at positions given by the angle θ as shown. Then

$$\begin{aligned} dM_{y=0} &= \left(\int_0^{\pi} (s\sin\theta)s\,d\theta\right) ks\,ds \\ &= 2ks^3\,ds, \\ M_{y=0} &= 2k\int_0^a s^3\,ds = \frac{ka^4}{2}. \end{aligned}$$

Therefore, $\overline{y} = \dfrac{ka^4}{2}\cdot\dfrac{3}{k\pi a^3} = \dfrac{3a}{2\pi}$. By symmetry, $\overline{x} = 0$. Thus, the centre of mass of the plate is $\left(0, \dfrac{3a}{2\pi}\right)$.

1. a) The nth bead extends from $x = (n-1)\pi$ to $x = n\pi$, and has volume

$$\begin{aligned} V_n &= \pi \int_{(n-1)\pi}^{n\pi} e^{-2kx} \sin^2 x \, dx \\ &= \frac{\pi}{2} \int_{(n-1)\pi}^{n\pi} e^{-2kx} (1 - \cos(2x)) \, dx \\ &\quad \text{Let } x = u + (n-1)\pi \\ &\quad dx = du \\ &= \frac{\pi}{2} \int_0^{\pi} e^{-2ku} e^{-2k(n-1)\pi} \big[1 - \cos(2u + 2(n-1)\pi)\big] \, du \\ &= \frac{\pi}{2} e^{-2k(n-1)\pi} \int_0^{\pi} e^{-2ku} (1 - \cos(2u)) \, du \\ &= e^{-2k(n-1)\pi} V_1. \end{aligned}$$

Thus $\dfrac{V_{n+1}}{V_n} = \dfrac{e^{-2kn\pi} V_1}{e^{-2k(n-1)\pi} V_1} = e^{-2k\pi}$, which depends on k but not n.

b) $V_{n+1}/V_n = 1/2$ if $-2k\pi = \ln(1/2) = -\ln 2$, that is, if $k = (\ln 2)/(2\pi)$.

c) Using the result of Example 4 in Section 7.1, we calculate the volume of the first bead:

$$\begin{aligned} V_1 &= \frac{\pi}{2} \int_0^{\pi} e^{-2kx} (1 - \cos(2x)) \, dx \\ &= \frac{\pi e^{-2kx}}{-4k} \bigg|_0^{\pi} - \frac{\pi}{2} \frac{e^{-2kx}(2\sin(2x) - 2k\cos(2x))}{4(1+k^2)} \bigg|_0^{\pi} \\ &= \frac{\pi}{4k}(1 - e^{-2k\pi}) - \frac{\pi}{4(1+k^2)}(k - ke^{-2k\pi}) \\ &= \frac{\pi}{4k(1+k^2)}(1 - e^{-2k\pi}). \end{aligned}$$

By part (a) and Theorem 1(d) of Section 6.1, the sum of the volumes of the first n beads is

$$\begin{aligned} S_n &= \frac{\pi}{4k(1+k^2)}(1 - e^{-2k\pi}) \\ &\quad \times \big[1 + e^{-2k\pi} + \big(e^{-2k\pi}\big)^2 + \cdots + \big(e^{-2k\pi}\big)^{n-1}\big] \\ &= \frac{\pi}{4k(1+k^2)}(1 - e^{-2k\pi}) \frac{1 - e^{-2kn\pi}}{1 - e^{-2k\pi}} \\ &= \frac{\pi}{4k(1+k^2)}(1 - e^{-2kn\pi}). \end{aligned}$$

Thus the total volume of all the beads is

$$V = \lim_{n\to\infty} S_n = \frac{\pi}{4k(1+k^2)} \text{ cu. units..}$$

12.
$$\begin{aligned} s &= \int_{\pi/6}^{\pi/4} \sqrt{1+\tan^2 x}\,dx \\ &= \int_{\pi/6}^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|\Big|_{\pi/6}^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\ &= \ln\frac{\sqrt{2}+1}{\sqrt{3}} \text{ units.} \end{aligned}$$

28. The area of the cone obtained by rotating the line $y = (h/r)x$, $0 \le x \le r$, about the y-axis is

$$\begin{aligned} S &= 2\pi \int_0^r x\sqrt{1 + (h/r)^2}\, dx = 2\pi \frac{\sqrt{r^2 + h^2}}{r} \frac{x^2}{2}\bigg|_0^r \\ &= \pi r \sqrt{r^2 + h^2} \text{ sq. units.} \end{aligned}$$