Math 121 Final Exam - Sample 1

Total time allowed: 3 hours

1. (10 points) Compute the following integral:

$$\int \frac{dx}{x^3 + x^2 + x}$$

2. (10 points) Verify that

$$S_{2n} = \frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3},$$

where T_n and M_n refer to the appropriate Trapezoid and Midpoint Rule approximations. Deduce that

$$S_{2n} = \frac{4T_{2n} - T_n}{3}$$

3. (10 points) Find the sum of the following series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-4}}{(2n-1)!}$$

4. (10 points) Evaluate the following limit:

$$\lim_{x \to 0} \frac{(x - \tan^{-1} x)(e^{2x} - 1)}{2x^2 - 1 + \cos(2x)}$$

5. (20 points) Let

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{2^{2k} k!}{(2k+1)!} x^{2k+1} \\ &= x + \frac{2}{3}x^3 + \frac{4}{3 \times 5}x^5 + \frac{8}{3 \times 5 \times 7}x^7 + \cdots \end{aligned}$$

   (a) Find the radius of convergence of this power series.
   (b) Show that $f'(x) = 1 + 2xf(x)$.
   (c) What is $\frac{d}{dx}\left(e^{-x^2} f(x)\right)$?
   (d) Express $f(x)$ in terms of an integral.

6. (10 points) A solid is 6 ft. high. Its horizontal cross-section at height z ft. above its base is a rectangle with length $2 + z$ ft. and width $8 - z$ ft. Find the volume of the solid.

7. (10 points) Find the length of the following curve from $x = \pi/6$ to $x = \pi/4$.

$$y = \ln \cos(x)$$

8. (20 points) A pyramid with square base, 4 m. on each side and four equilateral triangular faces, sits on the level bottom of a lake at a place where the lake is 10 m. deep. Find the total force of the water on each of the triangular faces.

9. (15 points) Sketch and find the area of the polar region R given by one leaf of the curve $r = \sin(3\theta)$.

10. (10 points) Evaluate, if possible, the limit of the followoing sequence:

$$a_n = \frac{(n!)^2}{(2n)!}$$

11. (5+5+10+5 points)
    (a) True or False:If neither $\{a_n\}$ not $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge.
    (b) Solve the following equation:

    $$\frac{dy}{dx} + 2\frac{y}{x} = \frac{1}{x^2}$$

    (c) Suppose you have a coin for which head and tail each occur 49% of the time, and it remains standing on its edge only 2% of the time. How much should you be willing to pay to play a game where you toss this coin and win $1 if it comes up head, $2 if it comes up tails, and $50 if it remains standing on its edge? Assume you will play the game any times and would like to at least break even.
    (d) State whether the following integral converges or diverges, and justify your claim:

    $$\int_0^\infty \frac{dx}{xe^x}$$