Math 105 - Practice Midterm 2 for Midterm 2 Solutions

This practice midterm may be harder and/or longer than the real midterm.
Not all question will be worth the same number of points.

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1. *Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.*
   See the last page of these solutions.

2. *A bank account has \$20,000 earning 5% interest compounded continuously. A pensioner uses the account to pay himself an annuity, drawing continuously at a \$2000 annual rate. How long will it take for the balance in the account to drop to zero?*

$$\frac{dx}{dt} = 0.05x - 2000 = 0.05(x - 40000), \qquad x(0) = 20000$$

$$\Rightarrow \quad \int \frac{1}{x-40000}dx = \int 0.05dt \quad \Rightarrow \quad \ln|x - 40000| = 0.05t + C_1.$$

$$\Rightarrow \quad \pm(x - 40000) = e^{0.05t+C_1} = e^{C_1}e^{0.05t}$$

$$\Rightarrow \quad x = 40000 + C_2e^{0.05t},$$

where C_2 can be any number (e^{C_1} can only be positive, but because of the $\pm$ we can have $C_2 = \pm e^{C_1}$).

$$20000 = x(0) = 40000 + C_2e^{0.05\cdot 0} = 40000 + C_2 \quad \Rightarrow \quad C_2 = -20000$$

$$\Rightarrow \quad x(t) = 40000 - 20000e^{0.05t} = 20000(2 - e^{0.05t}).$$

The balance will equal zero when

$$0 = 2 - e^{0.05t} \quad \Rightarrow \quad t = \frac{1}{0.05}\ln(2) = \boxed{20\ln(2)}\,(\approx 14 \text{ years}).$$

3. *Sketch the xy-trace, xz-trace, and yz-trace of the surface $z = 4y^2 - 9x^2$.*
   See the last page of these solutions.

4. *Evaluate the limit $\displaystyle\lim_{(x,y)\to(4,1)} \frac{x^2 - 4xy^4}{\sqrt{x} - 2y^2}$, or show that it doesn't exist.*

$$\lim_{(x,y)\to(4,1)} \frac{x^2 - 4xy^4}{\sqrt{x} - 2y^2} = \lim_{(x,y)\to(4,1)} \frac{x(x - 4y^4)}{\sqrt{x} - 2y^2} = \lim_{(x,y)\to(4,1)} \frac{x(\sqrt{x} - 2y^2)(\sqrt{x} + 2y^2)}{\sqrt{x} - 2y^2}$$

$$= \lim_{(x,y)\to(4,1)} x(\sqrt{x} + 2y^2) = 4(\sqrt{4} + 2\cdot 1^2) = 16.$$

We could also have multiplied by the conjugate $\sqrt{x} + 2y^2$ in the numerator and denominator.

5. *Consider the function* $f(x,y) = x^2 - 3y^2$.

   (a) *Calculate* f_x *and* f_y.

   $$f_x(x,y) = 2x, \qquad f_y(x,y) = -6y$$

   (b) *Find the rate of maximum increase when* $x = 3$, $y = 2$.
   The rate of maximum increase is $|\nabla f(3,2)|$, the length of the gradient vector, which is

   $$\nabla f(3,2) = \langle f_x(3,2), f_y(3,2)\rangle = \langle 2\cdot 3, -6\cdot 2\rangle = \langle 6, -12\rangle$$

   $$\Rightarrow \quad |\nabla f(3,2)| = \sqrt{6^2 + (-12)^2} = \sqrt{36+144} = \sqrt{180} = \boxed{6\sqrt{5}.}$$

   (c) *Sketch the level curve at height* $z = 4$. *Find the slope* $\frac{dy}{dx}$ *of the tangent line to this level curve at* $(x,y) = (4,2)$.
   See the last page of these solutions.

6. *Find the linear approximation for* $\sqrt{(3.06)^2 + (3.92)^2}$.
   We should use $z = f(x,y) = \sqrt{x^2+y^2}$ and $(a,b) = (3,4)$, and the linear approximation formula for either $L(x,y)$ or dz. For both we need

   $$f_x = \frac{x}{\sqrt{x^2+y^2}}, \qquad f_y = \frac{y}{\sqrt{x^2+y^2}},$$

   $$f(3,4) = \sqrt{3^2+4^2} = \sqrt{25} = 5, \quad f_x(3,4) = \frac{3}{5}, \quad f_y(3,4) = \frac{4}{5}.$$

   Using the linear approximation formula

   $$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

   we get the approximation

   $$L(3.06, 3.92) = \frac{3}{5}(3.06-3) + \frac{4}{5}(3.92-4) + 5 = \frac{3}{5}\cdot 0.06 + \frac{4}{5}\cdot(-0.08) + 5 = 0.036 - 0.064 + 5 = \boxed{4.972.}$$

   Note: on the midterm, you could leave $\frac{3}{5}\cdot 0.06 + \frac{4}{5}\cdot(-0.08) + 5$ *as your answer (or probably the calculation would be easier).*
   With the formula

   $$dz = f_x(a,b)dx + f_y(a,b)dy$$

   we get (with $dx = 3.06 - 3 = 0.06$, $dy = 3.92 - 4 = -0.08$)

   $$dz = \frac{3}{5}\cdot 0.06 + \frac{4}{5}\cdot(-0.08) = 0.036 - 0.064 = -0.028,$$

   so the approximation is

   $$\sqrt{(3.06)^2 + (3.92)^2} \approx f(3,4) + dz = 5 + (-0.028) = \boxed{4.972.}$$

   Note: the actual value is $\sqrt{(3.06)^2 + (3.92)^2} = 4.9729\cdots$.

7. *Find the critical points of* $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$, *and classify each one as a maximum, minimum or saddle point.*

$$f_x = -6x + 6y = 0, \qquad f_y = 6y - 6y^2 + 6x = 0$$

To solve these equations, we can get $y = x$ from the first equation, and plug that into the second:

$$\Rightarrow \quad 6y - 6y^2 + 6y = 0 \quad \Rightarrow \quad 0 = 12y - 6y^2 = 6y(2 - y) \quad \Rightarrow \quad y = 0, y = 2.$$

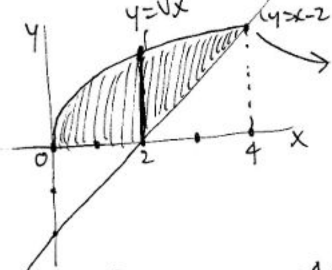
So the critical points are $(0, 0)$ and $(2, 2)$.
To classify them with the Second Derivative Test, we need the second partial derivatives and the discriminant:

$$f_{xx} = -6, \quad f_{yy} = 6 - 12y, \quad f_{xy} = 6$$

$$\Rightarrow \quad D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -6(6 - 12y) - 6^2 = -72 + 72y = 72(y - 1).$$

Then for the critical point $(0, 0)$ we have $D(0, 0) = -72 < 0$, so $\boxed{(0,0) \text{ is a saddle point.}}$.
For $(2, 2)$ we have $D(2, 2) = 72 > 0$, so we look at $f_{xx}(2, 2) = -6 < 0$, which tells us that $\boxed{(2,2) \text{ is a local maximum}}$.

1] area in first quadrant between $y=\sqrt{x}$, the x-axis and $y=x-2$

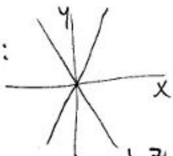


$$x-2=\sqrt{x} \Rightarrow (x-2)^2 = x$$
$$\Rightarrow x^2-5x+4=0$$
$$(x-4)(x-1)$$
$$\Rightarrow \underline{x=4}$$ (x=1 is no solution: $1-2 \neq \sqrt{1}$)

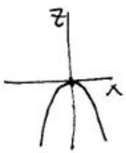
$$\Rightarrow A=\int_0^2 \sqrt{x}\,dx + \int_2^4 \left(\sqrt{x}-(x-2)\right)dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2+2x\right)\Big|_2^4$$
$$= \cancel{\frac{2}{3}2^{\frac{3}{2}}} + \left(\frac{2}{3}\cdot 8 \cancel{-\frac{1}{2}\cdot 16 + 2\cdot 4}\right) - \left(\cancel{\frac{2}{3}2^{\frac{3}{2}}} - \frac{1}{2}\cdot 4 + 2\cdot 2\right)$$
$$= \frac{16}{3}+6 = \boxed{\frac{34}{3}}.$$

3] $z=4y^2-9x^2$

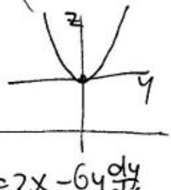
- xy-trace: $z=0 \Rightarrow 4y^2-9x^2=0 \Rightarrow y=\frac{3}{2}x$ or $y=-\frac{3}{2}x$
  $(2y-3x)(2y+3x)$ $\Rightarrow$ lines:



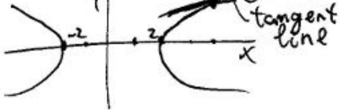
- xz-trace: $y=0 \Rightarrow z=-9x^2$
  $\Rightarrow$ parabola:



- yz-trace: $x=0 \Rightarrow z=4y^2 \Rightarrow$



5c] $4=x^2-3y^2 \Rightarrow$ hyperbola
($y=0 \Rightarrow x=\pm 2$)



Slope at (4,2):
implicit diff. $\Rightarrow 0=2x-6y\frac{dy}{dx}$
$$\Rightarrow \frac{dy}{dx}=\frac{x}{3y}$$
at (4,2): $\frac{dy}{dx}=\frac{4}{6}=\boxed{\frac{2}{3}}$