Solutions to Math 105 Practice Midterm2, Spring 2011

1. Short answer questions

(1) Let $u = \arctan x$ and $du = \frac{1}{x^2+1}dx$. Then

$$\int \frac{\arctan x}{x^2+1}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\arctan x)^2 + C.$$

So

$$\int_1^\infty \frac{\arctan x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{\arctan x}{x^2+1}\,dx = \lim_{b\to\infty}[\frac{1}{2}(\arctan x)^2]|_1^b$$
$$= \lim_{b\to\infty}\frac{1}{2}[(\arctan b)^2 - (\arctan 1)^2] = \frac{1}{2}[(\frac{\pi}{2})^2 - (\frac{\pi}{4})^2] = \boxed{\tfrac{3\pi^2}{32}}.$$

(2). The differential equation is separable. So we have

$$\frac{x}{x^2+5}\,dx = dt, \text{ then } \int \frac{x}{x^2+5}\,dx = \int dt$$

Let $u = x^2+5$ and $du = 2x\,dx$. Then

$$\int \frac{x}{x^2+5}dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C.$$

So we have

$$\frac{1}{2}\ln|u| = t + C,\ \ln|u| = 2t + 2C,\ |u| = Ce^{2t},\ x^2+5 = Ce^{2t},\ x^2 = Ce^{2t} - 5.$$

Since $x(0) = 1$, we have $1^2 = Ce^0 - 5$, that is $C = 6$. Thus $\boxed{x(t) = \sqrt{6e^{2t}-5}}$. (We do not take the branch $x = -\sqrt{6e^{2t}-5}$ because $x(0) = 1 > 0$.)

(3). The function $\frac{x^2y+xy^2}{y^3-x^3}$ is not well-define at $(0,0)$ and can not be simplified. We can apply "two path test". Let $y = mx$. Notice that $m \neq 1$ because $y = x$ is not in the domain. Then

$$\lim_{(x,y)\to(0,0)}\frac{x^2y+xy^2}{y^3-x^3} = \lim_{(x,y)\to(0,0)}\frac{x^2mx + x(mx)^2}{(mx)^3 - x^3} = \lim_{(x,y)\to(0,0)}\frac{x^3(m+m^2)}{x^3(m^3-1)} = \frac{m+m^2}{m^3-1}.$$

When $m = 0$, *Limit* $= 0$, when $m = 2$, *Limit* $= \frac{2+2^2}{2^3-1} = \frac{6}{7}$. So $\boxed{\text{the limit does not exist}}$.

(4). When $(x,y) = (6,1)$, $z = \sqrt{6-1^2-1} = 2$. So the level curve through $(6,1)$ has the equation $2 = \sqrt{x-y^2-1}$, that is $y^2 = x - 5$. $\boxed{\text{It is a parabola symmetric by } x\text{-axis with } x\text{-intersection } 5}$.

The level curve has the form $g(x,y)=0$, where $g(x,y)=y^2-x+5$. Since $g_x=-1$, and $g_y=2y$ then by differentiation implicitly we have

$$\frac{dy}{dx}=-\frac{g_x}{g_y}=\frac{1}{2y}.\ \text{Thus}\ \frac{dy}{dx}|_{(6,1)}=\frac{1}{2y}|_{(6,1)}=\boxed{\tfrac{1}{2}}.$$

2. First calculate the critical points: $f_x=4x^3-4y=0$, $f_y=4y-4x=0$. From the second equation we have $y=x$. Substitute into the first we have $x^3-x=x(x-1)(x+1)=0$. So $x=0,\ -1,\ 1$. We have three critical points $\boxed{(0,\ 0),\ (-1,\ -1)\text{ and }(1,\ 1)}$. We apply the second derivative test to classify the critical points.

$f_{xx}=12x^2$, $f_{yy}=4$ and $f_{xy}=-4$. Then for $(0,0)$, $f_{xx}=0$, $D(0,0)=0\cdot 4-(-4)^2=-16<0$. Thus $\boxed{(0,0)\text{ is a saddle point}}$. For $(1,1)$, $f_{xx}=12>0$, $D(1,1)=12\cdot 4-(-4)^2=48-16=32>0$. Thus $\boxed{(1,1)\text{ is a local minimum}}$. For $(-1,\ -1)$, $f_{xx}=12>0$, $D(-1,\ -1)=12\cdot 4-(-4)^2=48-16=32>0$. Thus $\boxed{(-1,\ -1)\text{ is also a local minimum}}$.

3.(a). The differential equation is $\boxed{y'(t)=y(t)0.05-A}$.
(b). By formula of present value, we have $\int_0^{25} A\cdot e^{-0.05t}\,dt=240,000$. Then

$$A\cdot\int_0^{25} e^{-0.05t}\,dt=A\cdot\frac{1}{-0.05}e^{-0.05t}|_0^{25}=A\cdot\frac{e^{-1.25}-1}{-0.05}=240,000.$$

Thus

$$A=240,000\cdot\frac{0.05}{1-e^{-1.25}}=\boxed{\tfrac{12,000}{1-e^{-1.25}}\approx 16818.6}.$$

(c). The total interest paid is $\boxed{16818.6\times 25-240,000\approx \$180465}$.

4. (a).

$$f_x=\frac{1}{2}(4-x^2-\frac{y^2}{4})^{-1/2}(-2x)=\frac{-x}{\sqrt{4-x^2-\frac{y^2}{4}}},$$

$$f_y=\frac{1}{2}(4-x^2-\frac{y^2}{4})^{-1/2}(-\frac{y}{2})=\frac{-y}{4\sqrt{4-x^2-\frac{y^2}{4}}}.$$

(b). Since $f_x(1,2)=-\frac{1}{\sqrt{2}}$, $f_y(1,2)=-\frac{1}{2\sqrt{2}}$ then the gradient is

$$\nabla f(1,2)=<-\frac{1}{\sqrt{2}},\ -\frac{1}{2\sqrt{2}}>.$$

It has length $\sqrt{(-\frac{1}{2})^2+(-\frac{1}{2\sqrt{2}})^2}=\frac{\sqrt{5}}{2\sqrt{2}}$. Then the unit vector giving direction of gradient is $\boxed{<-\frac{2}{\sqrt{5}},\ -\frac{1}{\sqrt{5}}>}$. Assume $<x,y>$ is the vector pointing the direction where there is no instantaneous change. Then $< x, y > \cdot < -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} >= 0$, that is $-\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y = 0$. $x = 1$, $y = -2$ is a solution. So $< 1, -2 >$ is the desired vector. It has length $\sqrt{1^2 + (-2)^2} = \sqrt{5}$. Then the unit vector is $\boxed{< \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} >}$.

(c). Notice that $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$ and $x = t$, $y = t^2 + 1$. Then by chain rule we have

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= \frac{-x}{\sqrt{4 - x^2 - \frac{y^2}{4}}} + \frac{-y}{4\sqrt{4 - x^2 - \frac{y^2}{4}}}(2t)\end{aligned}$$

At point $(1, 2, f(1, 2))$, we have $x = 1$, $y = 2$, $t = 1$. So the changing rate is

$$\begin{aligned}\frac{dz}{dt}|_{t=1} &= \frac{-x}{\sqrt{4 - x^2 - \frac{y^2}{4}}}|_{x=1,y=2} + \frac{-y}{4\sqrt{4 - x^2 - \frac{y^2}{4}}}(2t)|_{x=1,y=2,t=1} \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}} \cdot 4 = \boxed{-\sqrt{2}}.\end{aligned}$$

5. First sketch the graph of these two functions. Then calculate the intersection points. Let $x^3 - 4x = -x^2 + 2x$. We have $x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x - 2)(x + 3) = 0$. Then for $x \geq 0$, we have two intersection points $x = 0$ and $x = 2$. Also, one can figure out that $g(x)$ is larger on the interval $[0, 2]$. Thus

$$\begin{aligned}Area = \int_0^2 (-x^2 + 2x) - (x^3 - 4x)\, dx &= \int_0^2 (-x^3 - x^2 + 6x)\, dx \\ &= [-\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3x^2]_0^2 = \boxed{\frac{16}{3}}.\end{aligned}$$