Math 105 Final

1a. Determine whether the following improper integral is convergent or divergent. If it is convergent, compute the integral.

$$\int_0^5 \frac{1}{\sqrt{25-x^2}}\, dx.$$

1b. Let $F(x) = \int_{\sin(x)}^{\cos(x)} e^{-t^2}\, dt$. Compute $F'(x)$

1c. Numerical integration Use the Midpoint Rule with $n = 3$ to approximate the following integral:

$$\int_1^{2.5} (x-1)^2\, dx$$

1d. Determine whether or not the following limit exists:

$$\lim_{(x,y)\to(5,5)} \frac{x^2+y^2-2yx}{x-y}$$

1e. Consider the quadric surface defined by the equation

$$z = 4x^2 - 9y^2.$$

Draw level curves corresponding to the values $z = 0, \pm 1, 2$. Identify the quadric surface (i.e. is it a paraboloid, hyperboloid, or ellipsoid?)

2. Consider the following Demand and Supply curves:

$$D(q) = 6 - q \quad S(q) = q^2.$$

Find the equilibrium point (p_e, q_e), and compute the Consumers' and Producers' surplus.

3. A company estimates that the income produced at time t by its factory will equal $1000 - 50t$. Find the present value over the next ten years, assuming a 5% interest rate.

4. Evaluate the following indefinite integral:

$$\int \frac{x^2}{\sqrt{16 - x^2}}\, dx.$$

5. Find the area of the region in the first quadrant bounded by the curve $y = \sqrt{x}$ and the curve $y = x^3$.

6. Let X be a continuous random variable with the following probability density function:

$$f(x) = \frac{1}{21}x^2\, dx, \quad 1 \leq x < 4.$$

Find the corresponding cumulative distribution function, $F(x)$. Use $F(x)$ to compute the following probabilities: $P(2 \leq X)$, $P(X \leq 3)$.

7. A recently deceased person was found in a room, where the room's temperature was $17°C$. According to Newton's law of cooling, the the temperature of the body $y(t)$ at t hours after death satisfies the differential equation:

$$y' = k(17 - y),$$

for some constant k.

Assume that the body's temperature at the time of death is $37°C$. Further assume that the it is $27°C$ after 4 hours. Determine the constant k, and solve the differential equation to find $y(t)$.

8. Let $f(x, y) = \sqrt{x^3 + 4xy + y^2x + y^4 - 4}$.

   a Find $f(1, 2)$, $f_x(1, 2)$, and $f_y(1, 2)$.

   b. Approximate the change in f as x changes from 1 to 1.2 and y changes from 2 to 1.9.

   c. Now assume that $y = g(x)$ is a function of x defined implicitly by the equation

   $$f(x, y) = 5.$$

   Find the equation of the tangent line to the graph $y = g(x)$ at $(x, y) = (1, 2)$.

9. Using the method of Lagrange Multipliers, find the maximum and minimum of $f(x, y) = 3x^2 - 2y^2 + 2y$, with the constraint $x^2 + y^2 = 1$.

Solutions:

1a. The integral does converge, and is equal to $\frac{\pi}{2}$.

1b. $(-\sin(x))e^{-\cos^2(x)} - (\cos(x))e^{-\sin^2(x)}$.

1c. $\frac{35}{32}$

1d. The limit does exist, and is equal to 0

1e. The trace for $z = 0$ is a pair of lines, $2x = \pm 3y$.
The trace for $z = 1$ is the hyperbola $1 = \frac{x^2}{(\frac{1}{2})^2} - \frac{y^2}{(\frac{1}{3})^2}$
The trace for $z = -1$ is the hyperbola $1 = \frac{y^2}{(\frac{1}{3})^2} - \frac{x^2}{(\frac{1}{2})^2}$
The trace for $z = 2$ is the hyperbola $1 = \frac{x^2}{(\frac{1}{\sqrt{2}})^2} - \frac{y^2}{(\frac{1}{3\sqrt{2}})^2}$. The quadric surface is a hyperbolic paraboloid.

2. $(p_e, q_e) = (2, 4)$ Producers surplus is $\frac{16}{3}$. Consumers' surplus is 2.

3. $\int_0^{10}(1000 - 50t)e^{-.05t}\,dt = 6065.3$

4. $8\arcsin(\frac{x}{4}) - \frac{1}{2}x\sqrt{16 - x^2}$.

5. $\frac{5}{12}$.

6. $F(x) = \frac{1}{63}(x^3 - 1)$.
$P(2 \leq X) = 1 - F(2) = \frac{48}{63}$
$P(X \leq 3) = F(3) = \frac{80}{63}$.

7. $k = \frac{1}{4}\ln(2)$.
$y(t) = 17 + 20e^{-\frac{1}{4}\ln(2)t}$.

8a. $f(1, 2) = 5$
$f_x(1, 2) = \frac{3}{2}$
$f_y(1, 2) = \frac{20}{5} = 4$

8b. $df = -\frac{1}{10}$.

8c. $y - 2 = -\frac{3}{8}(x - 1)$

9. Solutions are $(0, 1)$, $(0, -1)$, $(\frac{\sqrt{24}}{5}, \frac{1}{5})$, and $(-\frac{\sqrt{24}}{5}, \frac{1}{5})$.

$$f(0, 1) = 0$$
$$f(0, -1) = -4$$
$$f(\pm\frac{\sqrt{24}}{5}, \frac{1}{5}) = \frac{16}{5}.$$

So $(0, -1)$ is the minimum and $(\pm\frac{\sqrt{24}}{5}, \frac{1}{5})$ are both maximum.