Math 105 Assignment 4 Solutions

1. (10 points) Evaluate

$$\int \frac{x^{18}}{(49-x^2)^{\frac{21}{2}}}dx$$

Solution: The denominator suggests a trigonometric substitution involving the sin function. Perform the substitution $x = 7\sin(\theta)$, $dx = 7\cos(\theta)$ to get

$$\int \frac{x^{18}}{(49-x^2)^{\frac{21}{2}}}dx = \int \frac{(7\sin(\theta))^{18}}{(49-(7\sin(\theta))^2)^{\frac{21}{2}}} \cdot 7\cos(\theta)d\theta = \int \frac{(7\sin(\theta))^{18}}{(49\cos^2(\theta))^{\frac{21}{2}}} \cdot 7\cos(\theta)d\theta$$

$$= \int \frac{(7\sin(\theta))^{18}}{(7\cos(\theta))^{21}} \cdot 7\cos(\theta)d\theta = \int \tan^{18}(\theta) \cdot \frac{1}{49}\sec^2(\theta)d\theta$$

We can take care of this last expression using the substitution $u = \tan(\theta)$, $du = \sec^2(\theta)dx$ to get

$$\frac{1}{49}\int \tan^{18}(\theta)\sec^2(\theta)du = \frac{1}{49}\int u^{18}du = \frac{1}{49}\cdot\frac{u^{19}}{19} + C = \frac{1}{49}\cdot\frac{\tan^{19}(\theta)}{19} + C$$

To substitute our original variable $x = 7\sin(\theta)$ back in, we need to change this to an expression in terms of sin. Rewriting,

$$\frac{1}{49}\cdot\frac{\tan^{19}(\theta)}{19} + C = \frac{1}{931}\cdot\frac{7\sin^{19}(\theta)}{7\cos^{19}(\theta)} + C = \frac{1}{931}\cdot\frac{\sin^{19}(\theta)}{(\sqrt{49-49\sin^2(\theta)})^{\frac{19}{2}}} + C = \frac{1}{931}\cdot\frac{x^{19}}{(49-x^2)^{\frac{19}{2}}} + C$$

Our end result is then

$$\int \frac{x^{18}}{(49-x^2)^{\frac{21}{2}}}dx = \frac{1}{931}\cdot\frac{x^{19}}{(49-x^2)^{\frac{19}{2}}} + C$$

2. (15 points) Evaluate

$$\int \frac{x^3 - 4}{x^2 - 2x - 3} dx$$

Solution: We use partial fractions to simplify the integrand.

Since the degree of the numerator $x^3 - 4$ is not strictly less than the degree of the denominator $x^2 - 2x - 3$, we need to do long division first.

Divide the leading term of the numerator, x^3, by the leading term of the denominator, x^2. The result of this is x, so we subtract $x(x^2 - 2x - 3)$ from the numerator.

$(x^3 - 4) - x(x^2 - 2x - 3) = 2x^2 + 3x - 4$

Repeating this with $2x^2 + 3x - 4$, the result of dividing the leading term $2x^2$ by x^2 is 2, so we subtract $2(x^2 - 2x - 3)$.

$(2x^2 + 3x - 4) - 2(x^2 - 2x - 3) = 7x + 2$, so $x^4 - 4 = (x + 2)(x^2 - 2x - 3) + (7x + 2)$.

This means our integral is

$$\int \frac{x^3 - 4}{x^2 - 2x - 3} dx = \int \left((x + 2) + \frac{7x + 2}{x^2 - 2x - 3} \right) dx$$

We use partial fractions to take care of $\int \frac{7x+2}{x^2-2x-3} dx$. First note that the denominator factors as $x^2 - 2x - 3 = (x - 3)(x + 1)$

To do this, we want numbers A, B with $\frac{7x+2}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$.

Clearing denominators gives $7x+2 = (x-3)(x+1)\frac{A}{x-3} + (x-3)(x+1)\frac{B}{x+1} = (x-3)A + (x+1)B = (A + B)x + (A - 3B)$

The coefficient of x on the left is 7 and the coefficient of x on the right is $A + B$, so we get the equation $7 = A + B$.

The constant term on the left is 2, and the constant term on the right is $3A - 2B$, so we get the equation $2 = A - 3B$.

Adding three times the first equation to the second gives $23 = 3 \cdot 7 + 2 = 3A + A + 3B - 3B = 4A$, so $A = \frac{23}{4}$, and plugging back into the original equation, we get $B = \frac{5}{4}$.

Now we have

$$\int \frac{7x+1}{x^2-2x-3}dx = \int \left(\frac{\frac{23}{4}}{x-3} + \frac{\frac{5}{4}}{x-1} \right) dx = \frac{23}{4}\ln(x-3) + \frac{5}{4}\ln(x-1) + C$$

For our original integral, we now have

$$\int \frac{x^3-4}{x^2-2x-3}dx = \int (x+2)dx + \int \frac{7x+2}{x^2-2x-3}dx = \frac{1}{2}x^2 + 2x + \frac{23}{4}\ln(x-3) + \frac{5}{4}\ln(x-1) + C$$

3. (15 points) Consider the integral

$$\text{I} = \int_0^1 \sqrt{1-x^2}dx.$$

(a) Find I (use geometry). (b) Find an approximation of I using Midpoint rule and Trapezoid rule with $n = 4$. (c) For both Midpoint and Trapezoid rules, calculate the absolute error between the estimate and the true value.

Solution: (a) From 0 to 1, the function traces a quarter circle of radius 1. The area underneath is therefore a quarter of the area of a circle with radius one, so $I = \frac{\pi}{4}$.

(b) First we need to partition the interval into $n = 4$ parts. The partition has grid points $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$. The width of the intervals is $\Delta x = \frac{1}{4}$.

Midpoint Rule: The midpoints of the intervals $[x_{k-1}, x_k]$ for $k = 1, 2, 3, 4$ are $m_1 = \frac{1}{8}, m_2 = \frac{3}{8}, m_3 = \frac{5}{8}, m_4 = \frac{7}{8}$.

Evaluating the function $f(x) = \sqrt{1-x^2}$ at these points, we have $f(m_1) = \frac{\sqrt{63}}{8}, f(m_2) = \frac{\sqrt{55}}{8}, f(m_3) = \frac{\sqrt{39}}{8}, f(m_4) = \frac{\sqrt{15}}{8}$.

The midpoint rule approximation is

$$M(n) = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x = \sum_{k=1}^{n} f(m_k)\Delta x = \frac{1}{4}(\frac{\sqrt{63}}{8} + \frac{\sqrt{55}}{8} + \frac{\sqrt{39}}{8} + \frac{\sqrt{15}}{8}) \approx 0.795982$$

The absolute error is $\left|\frac{\pi}{4} - M(n)\right| \approx 0.010583$, and the relative error is $\dfrac{\left|\frac{\pi}{4} - M(n)\right|}{\frac{\pi}{4}} \approx 0.013475.$

Trapezoid Rule: Evaluating the function f at the grid points, $f(x_0) = f(0) = 1, f(x_1) = f(\frac{1}{4}) = \frac{\sqrt{15}}{4}, f(x_2) = f(\frac{1}{2}) = \frac{\sqrt{3}}{2}, f(x_3) = f(\frac{3}{4}) = \frac{\sqrt{7}}{4}, f(x_4) = f(1) = 0$.

The trapezoid rule approximation is

$$T(n) = \frac{f(x_0)}{2} + \sum_{k=1}^{n-1} f(x_k) + \frac{f(x_n)}{2} = \frac{1}{2} + \frac{\sqrt{15}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{7}}{4} + \frac{0}{2} \approx 0.748927$$

The absolute error is $\left|\frac{\pi}{4} - T(n)\right| \approx 0.036471$, and the relative error is $\dfrac{\left|\frac{\pi}{4} - T(n)\right|}{\frac{\pi}{4}} \approx 0.046437.$