

## Math 105 Assignment 7 Solutions

1. Compute

$$u_{xx} + u_{yy} \quad \text{if} \quad u(x, y) = \ln(x^2 + y^2).$$

**Solution:** First we use the chain rule to compute

$$u_x = \frac{\partial}{\partial x} \ln(x^2 + y^2) = \frac{1}{x^2 + y^2} \cdot 2x = \frac{2x}{x^2 + y^2}$$
$$u_y = \frac{\partial}{\partial y} \ln(x^2 + y^2) = \frac{1}{x^2 + y^2} \cdot 2y = \frac{2y}{x^2 + y^2}$$

Next we use the quotient rule to find the second partials:

$$u_{xx} = \frac{\partial}{\partial x} \frac{2x}{x^2 + y^2} = \frac{(x^2 + y^2) \cdot 2 - 2x \cdot 2x}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$
$$u_{yy} = \frac{\partial}{\partial y} \frac{2y}{x^2 + y^2} = \frac{(x^2 + y^2) \cdot 2 - 2y \cdot 2y}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$\text{Thus we have } u_{xx} + u_{yy} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0.$$

2. An ant sits on an overturned bowl shaped like the hemisphere

$$z = \sqrt{1 - x^2 - y^2}.$$

If it starts crawling directly above the curve  $x = y = e^{-t}$ ,  $t \geq \frac{1}{2}$  find the rate at which it will be gaining or losing height when it passes the point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ . Is it going uphill or downhill at that point?

**Solution:** We have the  $z$ -coordinate for the point on the hemisphere lying above the curve  $x = y = e^{-t}$  as a curve in the parameter  $t$ :  $z = \sqrt{1 - 2e^{-2t}}$ .

The rate of change of the  $z$ -coordinate is then

$$z'(t) = \frac{2e^{-2t}}{\sqrt{1 - 2e^{-2t}}}$$

The point  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$  occurs along the curve when  $t = -\ln(\frac{1}{2})$ .

Plugging this in,

$$z'(-\ln(\frac{1}{2})) = \frac{2 \cdot \frac{1}{4}}{\sqrt{1 - \frac{1}{4}}} = \frac{1}{\sqrt{3}}$$

Since the rate of change of the  $z$ -coordinate is positive, the ant is going uphill at this point.

3. Consider the production function of an enterprise

$$f(x, y) = 60x^{\frac{3}{4}}y^{\frac{1}{4}},$$

which gives the number of units of goods produced when utilizing  $x$  units of labor and  $y$  units of capital. The enterprise currently employs 81 units of labor and invests 16 units of capital.

- (a) If the capital is held fixed at its current value, and the amount of labor increased by one unit, find the approximate change in the quantity of goods produced. This quantity is known as the *marginal productivity of labor*.

**Solution:** The marginal productivity of labor is the partial derivative of the production function with respect to  $x$ . We have  $\frac{\partial}{\partial x}f(x, y) = 45x^{-\frac{1}{4}}y^{\frac{1}{4}}$ .

At the point  $(81, 16)$ , the marginal productivity of labor is

$$\frac{\partial}{\partial x}f(81, 16) = 45 \cdot 81^{-\frac{1}{4}} \cdot 16^{\frac{1}{4}} = 30$$

- (b) Similarly, holding the amount of labor fixed at its current value, determine the approximate increase in productivity generated by an increase in capital by one unit. This is called the *marginal productivity of capital*.

**Solution:** The marginal productivity of capital is the partial derivative of the production function with respect to  $y$ . We have  $\frac{\partial}{\partial y}f(x, y) = 15x^{\frac{3}{4}}y^{-\frac{3}{4}}$ .

At the point  $(81, 16)$ , the marginal productivity of capital is

$$\frac{\partial}{\partial y}f(81, 16) = 15 \cdot 81^{\frac{3}{4}} \cdot 16^{-3/4} = 50.625$$