Math 105, Spring 2011
Midterm 1 Solutions

Question: Compute the following integral:

$$\int \frac{e^{2x}}{(e^x+1)(e^x+2)}dx$$

Solution. The first step is to make the substitution

$$u = e^x \quad \text{so that} \quad du = e^x dx.$$

Under this change of variable, the integral transforms as follows:

$$\int \frac{e^{2x}}{(e^x+1)(e^x+2)}dx = \int \frac{u\,du}{(u+1)(u+2)}.$$

Note that the integrand in the last integral is a rational function (ratio of two polynomials), whose numerator is of lower degree than the denominator. Further the denominator is a product of two simple linear factors, so we should simplify the integrand using partial fractions. Accordingly, we write

$$\frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}, \qquad \text{or} \qquad u = A(u+2) + B(u+1)$$

where A and B are constants to be determined. Setting $u = -1$ and -2 respectively, we obtain $A = -1$ and $B = 2$. Thus

$$\begin{aligned}
\int \frac{e^{2x}}{(e^x+1)(e^x+2)}dx &= \int \frac{u\,du}{(u+1)(u+2)} \\
&= \int \left[\frac{2}{u+2} - \frac{1}{u+1}\right] du \\
&= 2\ln(u+2) - \ln(u+1) + C \\
&= 2\ln(e^x+2) - \ln(e^x+1) + C \\
&= \ln\left[\frac{(e^x+2)^2}{(e^x+1)}\right] + C.
\end{aligned}$$

□

Question: Compute the following integral:

$$\int_0^1 \arctan t \; dt$$

Solution. We apply integration by parts with $u = \arctan t$ and $v' = 1$, so that $u' = (t^2+1)^{-1}$ and $v = t$. This gives

$$\int \arctan t\,dt = uv - \int vu' = t\arctan t - \int \frac{t\,dt}{t^2+1}. \tag{1}$$

In order to compute the last integral, we substitute $u = t^2 + 1$, so that $du = 2tdt$ and

$$\int \frac{t\,dt}{t^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(t^2+1) + C. \tag{2}$$

Combining (1) and (2) we obtain

$$\int_0^1 \arctan t\,dt = \left[t\arctan t - \frac{1}{2}\ln(t^2+1)\right]_{t=0}^{t=1} = \arctan(1) - \frac{1}{2}\ln 2 + \frac{1}{2}\ln(1) = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

□

Question: Find the net area and the total area between the graph of the function $f(x) = 2\sin x$ and the x-axis on the interval $[-\frac{\pi}{2}, \pi]$.

Solution. The function $f(x) = 2\sin x$ is positive in $(0, \pi)$ and negative in $(-\pi/2, 0)$. We compute the signed areas of the positive and negative parts separately:

$$A_+ = \int_0^\pi 2\sin x\,dx = -2\cos x\Big|_{x=0}^{x=\pi} = 4,$$

$$A_- = \int_{-\pi/2}^0 2\sin x\,dx = -2\cos x\Big|_{x=-\pi/2}^{x=0} = -2.$$

So the net area is $A_+ + A_- = 4 - 2 = 2$, while the total area is $A_+ - A_- = 4 + 2 = 6$.

□

Question: Give short answers to the following questions.

(a) **Using Riemann sums, identify the following limit as a definite integral but do not evaluate the integral:**

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\cos\left(\frac{\pi k}{n}\right).$$

Solution 1. The sum above is a right Riemann sum with

$$x_k = \frac{k}{n},\ 1 \le k \le n, \qquad \text{so that} \qquad \Delta x = \frac{1}{n}$$

Thus $a = x_0 = 0$ and $b = x_n = 1$. Further

$$f(x_k) = f\left(\frac{k}{n}\right) = \cos\left(\pi\frac{k}{n}\right) = \cos(\pi x_k),$$

so $f(x) = \cos(\pi x)$. Thus

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\cos\left(\frac{\pi k}{n}\right) = \int_a^b f(x)\,dx = \int_0^1 \cos(\pi x)\,dx.$$

□

Solution 2. Alternatively, you could also have thought of the above sum as a right Riemann sum with

$$x_k = \frac{k\pi}{n},\ 1 \le k \le n \quad \text{in which case} \quad \Delta x = \frac{\pi}{n}, a = x_0 = 0,\ b = x_n = \pi.$$

In this case

$$\sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{\pi k}{n}\right) = \sum_{k=1}^{n} \frac{\pi}{n} \frac{\cos\left(\frac{\pi k}{n}\right)}{\pi} = \sum_{k=1}^{n} \Delta x f(x_k)$$

for

$$f(x_k) = f(\pi \frac{k}{n}) = \frac{1}{\pi} \cos\left(\frac{\pi k}{n}\right).$$

This means that

$$f(x) = \frac{1}{\pi} \cos x.$$

So, an alternative answer to the question is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{\pi k}{n}\right) = \int_a^b f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \cos x\, dx.$$

□

(b) **Find the following derivative in terms of f:**

$$\frac{d}{dx} \int_{\cos(x^2)}^{0} f(t)\ dt$$

Solution. By the fundamental theorem of calculus,

$$\begin{aligned} \frac{d}{dx} \int_{\cos(x^2)}^{0} f(t)\ dt &= -\frac{d}{dx} \int_0^{\cos(x^2)} f(t)\ dt \\ &= -f\left(\cos(x^2)\right) \frac{d}{dx}\left[\cos(x^2)\right] \\ &= f(\cos(x^2)) 2x \sin(x^2). \end{aligned}$$

□

(c) **Find the trapezoidal rule approximation of**

$$\int_{-1}^{2} 2x^2\ dx$$

with $n = 3$.

Solution. The interval $[-1, 2]$ has to be split into three equal pieces, so the endpoints of the subintervals are

$$x_0 = -1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad \text{hence } \Delta x = 1.$$

Also,

$$f(x_0) = 2, \quad f(x_1) = 0, \quad f(x_2) = 2, \quad f(x_3) = 8.$$

By the trapezoidal rule formula:

$$T(3) = \Delta x \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \frac{1}{2} f(x_3)\right] = 1 + 0 + 2 + 4 = 7.$$

□