MIDTERM 2 SOLUTIONS

1. Consider the following function:

$$f(x,y) = x^4 + y^3 - 4x - 12y + 10$$

(a) Find *all* the critical points of f.

Solution. We solve the equations

$$f_x(x,y) = 4x^3 - 4 = 0, \text{ and}$$
$$f_y(x,y) = 3y^2 - 12 = 0.$$

The first equations yields $x = 1$ while the second one has two solutions $y = \pm 2$. Thus there are two critical points $(1, 2)$ and $(1, -2)$. □

(b) For any one of the critical points you found in part (a), determine whether it is a local maximum, local minimum or a saddle point.

Solution. We apply the second derivative test. For this we need to compute the quantities

$$f_{xx} = 12x^2, \quad f_{yy} = 6y, \quad f_{xy} = 0 \quad \text{and} \quad D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 72x^2y$$

at the critical points. We observe that $D(1, -2) = -144 < 0$, which means that $(1, -2)$ is a saddle point. On the other hand, $D(1, 2) = 144 > 0$ and $f_{xx}(1, 2) = 12 > 0$, so by the second derivative test, $(1, 2)$ is a local minimum. □

2. (a) Draw the graphs of the two curves

$$y = x^2 + 1 \quad \text{and} \quad y = 4x - 2$$

in the (x, y)-plane and shade the region that is bounded by these curves.

(b) Find the area of the shaded region in part (a).

Solution. We solve for the points of intersection of the two curves. The x coordinates of the points of intersection are obtained from the equation

$$x^2 + 1 = 4x - 2 \qquad \text{i.e.,} \qquad x^2 - 4x + 3 = 0, \qquad \text{so that } x = 1, 3.$$

Since the region is bounded above by the straight line $y = 4x - 2$ and below by the parabola $y = x^2 + 1$, its area is given by

$$\begin{aligned}\int_1^3 \left[(4x-2) - (x^2+1)\right] dx &= \int_1^3 (4x - x^2 - 3)\, dx \\ &= \left[2x^2 - \frac{x^3}{3} - 3x\right]_{x=1}^{x=3} \\ &= (18 - 9 - 9) - (2 - \frac{1}{3} - 3) = \frac{4}{3}.\end{aligned}$$

□

3. The productivity of a company is given by

$$f(x,y) = 50\sqrt{x^2 + y^2},$$

where x and y are the amounts of labor and capital respectively. The company currently employs 4 units of labor and invests 3 units of capital.

(a) Find $f_x(4, 3)$ and $f_y(4, 3)$.

Solution.

$$f_x(x, y) = \frac{50x}{\sqrt{x^2 + y^2}} = 40,$$
$$f_y(x, y) = \frac{50y}{\sqrt{x^2 + y^2}} = 30.$$

□

(b) Determine the maximum possible rate at which the company could increase its productivity from its current values $x = 4$, $y = 3$.

Solution. The maximum rate of increase is given by

$$|\nabla f(4, 3)| = \sqrt{(40)^2 + (30)^2} = 50.$$

□

(c) Suppose that starting from $(4, 3)$, the labor and capital investments of the company evolve with time t according to the relation

$$x(t) = 4t, \qquad y(t) = 3t.$$

Find the rate of change in productivity with respect to t at the point $(4, 3)$. Will this rate change as time goes on?

Solution. Since

$$z(t) = f(x(t), y(t)) = 50\sqrt{(4t)^2 + (3t)^2} = 250t,$$

therefore $z'(t) = 250$. The rate of change in productivity is 250, which is independent of t, and hence does not change as time goes on. □

4. Give short answers to the following questions.

(a) A retirement account $A(t)$ has an initial amount of \$18,000. The account collects interest compounded continuously at the annual rate of 5%, and each year \$1000 is continuously withdrawn. Set up a differential equation that $A(t)$ satisfies, and specify the initial condition on $A(t)$. *Do not solve the differential equation. Solving the equation, even if correct, will not receive any credit.*

Solution. The differential equation for $A(t)$ is

$$A'(t) = 0.05A(t) - 1000,$$

with initial condition

$$A(0) = 18,000.$$

□

(b) Draw two separate diagrams for the two parts of the question below.

i. Sketch the level curve of the function $f(x, y) = xy$ at height 1.

ii. Sketch the level curve of the function $f(x, y) = xy$ at height -2.

(c) Establish whether the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 2y^4}{x^2 + y^4}$$

exists. If it does, find the limit. If not, explain why not.

Solution. Along the path $x = 0$,

$$\frac{x^2 + 2y^4}{x^2 + y^4} = \frac{2y^4}{y^4} = 2,$$

while along the path $y = 0$,

$$\frac{x^2 + 2y^4}{x^2 + y^4} = \frac{x^2}{x^2} = 1.$$

Since the function approaches two different values along two different paths converging to $(0, 0)$, the limit does not exist. $\square$