

# APPLICATION OF THE INTEGRAL I: CONSUMER AND PRODUCER SURPLUS

## 1. SUPPLY AND DEMAND

One of the most fundamental economic models is the law of supply and demand for a certain product (milk, bread, fuel etc.) or service (transportation, health care, education etc.) in a free-market environment. In this model the quantity of a certain item produced and sold is described by two curves, called the supply and demand curves of the item.

**Definition 1.1.** *The **supply function or supply curve** gives the quantity of an item that producers will supply at any given price. The **demand function or demand curve** gives the quantity that consumers will demand at any given price.*

We will denote the price per unit by  $p$  and the quantity supplied or demanded at that price by  $q$ . As is the convention in economics, we will always write  $p$  as a function of  $q$ . Thus the supply curve will be denoted by the formula

$$p = S(q)$$

and represented by a graph where the  $x$  and  $y$  axes correspond to  $q$  and  $p$  values respectively. Similarly, we will use

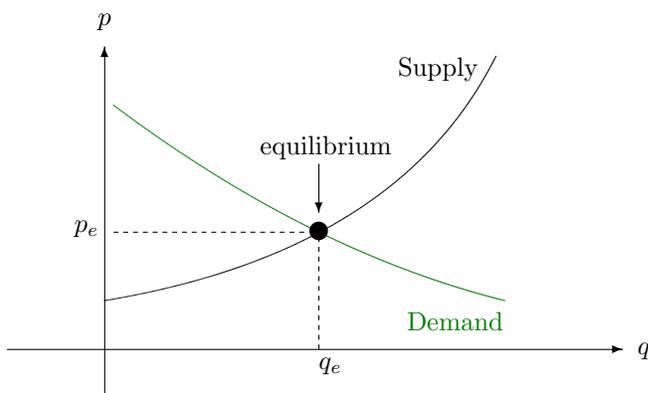
$$p = D(q)$$

to denote the demand curve.

As you might expect, the supply function  $S$  is increasing – the higher the price, the more the producers will supply. The demand function  $D$  is decreasing – the higher the price, the less the consumers will buy.

**Definition 1.2.** *The **point of intersection**  $(q_e, p_e)$  of the supply and demand curves is called the **market equilibrium point**. The numbers  $q_e$  and  $p_e$  are termed **equilibrium quantity** and **equilibrium price** respectively.*

The economic significance of the market equilibrium is the following: consider the case of bread. As long as  $p < p_e$ , the demand for bread exceeds its supply, pushing up the price until it reaches equilibrium price  $p_e$ . At this point, the quantity supplied is equal to the quantity demanded which is the equilibrium quantity. Conversely, if the price exceeds equilibrium, the supply of bread exceeds demand, bringing the price down (See figure below).



## 2. SURPLUS, RIEMANN SUMS AND INTEGRALS

In an ideal free market both consumers and producers gain by buying and selling at the equilibrium price. It is easy to understand this in principle, but the goal of this section is to compute exactly how much the consumers gain by buying at the equilibrium price rather than at a higher price.

We first compute the total amount spent by the consumers if everyone buys at the equilibrium price  $p_e$ . In that case  $q_e$  units are supplied and bought, and the total amount spent is the number of units bought times the price per unit, i.e.,

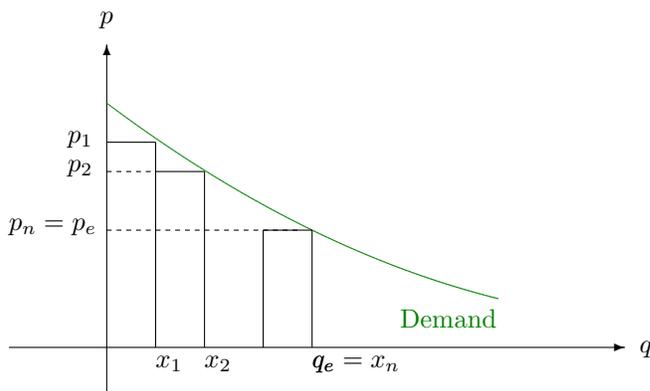
$$(2.1) \quad \text{total amount spent at equilibrium price} = p_e q_e.$$

Next, let us compute the total amount that would be spent if every consumer paid the maximum price that each is willing to pay. Divide the interval  $[0, q_e]$  into  $n$  subintervals, each of length  $\Delta x = q_e/n$ , with endpoints

$$x_0 = 0, x_1 = \frac{q_e}{n}, x_2 = \frac{2q_e}{n}, \dots, x_n = \frac{nq_e}{n} = q_e.$$

Consider the first interval  $[0, x_1]$ , i.e., suppose that only  $x_1$  units had been available. Then the selling price per unit could have been set at  $D(x_1)$  dollars and  $x_1$  units sold. Of course, at this price it would have been impossible to sell any more. The total expenditure from buying these first  $x_1$  units of commodity is therefore

$$(\text{price per unit}) \times (\text{number of units}) = D(x_1)\Delta x \text{ dollars.}$$



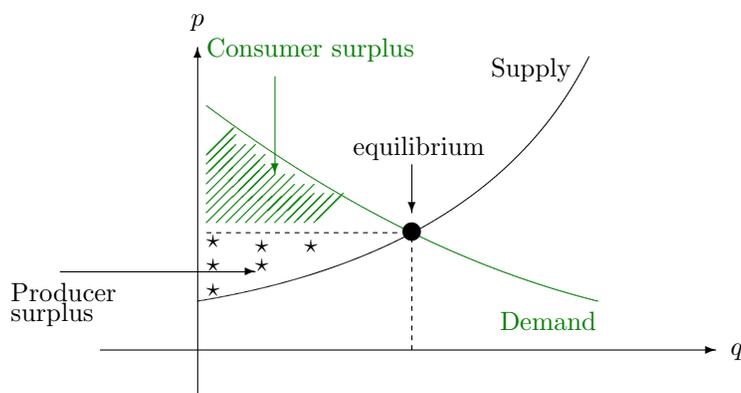
After selling the first  $x_1$  units, suppose that more units become available, so that now a total of  $x_2$  units have been produced. Setting the price at  $D(x_2)$ , the remaining  $x_2 - x_1 = \Delta x$  units can be sold, yielding  $D(x_2)\Delta x$  dollars. Note that each group of buyers paid as much for the commodity as it was worth to them. Continuing this process of price discrimination, the total amount of money paid by consumers willing to pay at least  $p_e$  is approximately equal to

$$(2.2) \quad D(x_1)\Delta x + D(x_2)\Delta x + \cdots + D(x_n)\Delta x.$$

The sum in (2.2) is a Riemann sum for the integral  $\int_0^{q_e} D(q) dq$ , and the sum gets closer and closer to the integral as  $n$  gets larger. At the same time, the sum gives better and better approximations to the total expenditure. The conclusion is therefore

$$(2.3) \quad \text{Total amount paid at maximum prices} = \int_0^{q_e} D(q) dq.$$

The quantity in (2.3) is the area under the demand curve from  $q = 0$  to  $q = q_e$ . As the figure shows, it is greater than  $p_e q_e$ , which is the area of the rectangle with sides  $[0, q_e]$  and  $[0, p_e]$ , and which according to the formula (2.1) represents the total amount spent by consumers at the equilibrium price. The difference between these two areas (2.3) - (2.1) represents the total that consumers save by buying at equilibrium price.



This is called the **consumer surplus** for this product (See picture above). To summarize

$$(2.4) \quad \text{Consumer surplus} = \int_0^{q_e} D(q) dq - p_e q_e = \int_0^{q_e} [D(q) - p_e] dq.$$

A similar analysis (which you should try out) shows that the producers also gain by trading at the equilibrium price. Their gain called **producer surplus** is given by the following quantity

$$(2.5) \quad \text{Producer surplus} = p_e q_e - \int_0^{q_e} S(q) dq = \int_0^{q_e} [p_e - S(q)] dq.$$

## 3. EXAMPLE

For a certain item the demand curve is

$$p = D(q) = \frac{20}{q+1}$$

and the supply curve is

$$p = S(q) = q + 2.$$

Find the equilibrium price and equilibrium quantity. Then compute the consumer and producer surplus.

**Solution.** To find the equilibrium quantity, we let  $D(q) = S(q)$  to obtain

$$\frac{20}{q+1} = q + 2.$$

Clearing the denominator gives  $20 = (q+1)(q+2)$ , which simplifies to  $q^2 + 3q - 18 = 0$ . The positive solution gives the equilibrium quantity  $q_e = 3$ , and the equilibrium price is  $p_e = 5$ .

We compute consumer and producer surplus using formulae (2.4) and (2.5) above:

$$\begin{aligned} \text{CS} &= \int_0^{q_e} D(q) \, dq - p_e q_e \\ &= \int_0^3 \frac{20}{q+1} \, dq - (5)(3) \\ &= 20 \ln(q+1) \Big|_0^3 - 15 \\ &= 20 \ln 4 - 15 \\ &\approx 12.73. \end{aligned}$$

Similarly

$$\begin{aligned} \text{PS} &= p_e q_e - \int_0^{q_e} S(q) \, dq \\ &= (5)(3) - \int_0^3 (q+2) \, dq \\ &= 15 - \left( \frac{q^2}{2} + 2q \right) \Big|_0^3 \\ &= 15 - \left( \frac{9}{2} + 6 \right) = 4.50. \end{aligned}$$

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