

Math 105 Week 9

March 7, 2011

1. LESSON PLAN

This week we will cover the following concepts:

- **gradient vector and its interpretations**, and
- **local maximum/minimum of functions of two variables**.

This material is contained in sections 12.6, 12.7 and 12.8 of the textbook, but not all the material in these sections will be relevant for us. Only the topics described below are in the syllabus.

Before launching into these sections, it may be helpful to refresh your memory regarding the following definitions:

- the *length* $|\langle a, b \rangle|$ of a vector $\langle a, b \rangle$:

$$|\langle a, b \rangle| = \sqrt{a^2 + b^2}.$$

- The *direction* \mathbf{u} of a vector $\langle a, b \rangle$:

$$\mathbf{u} = \left\langle \frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right\rangle.$$

The direction vector \mathbf{u} is often called a *unit vector* in the direction of $\langle a, b \rangle$, since it has length one.

- Two vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are orthogonal or perpendicular if

$$v_1 w_1 + v_2 w_2 = 0.$$

1.1. Sections 12.6 and 12.7. The *gradient vector* of f , denoted ∇f , is the vector of first partial derivatives of f , i.e.,

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

(See definition on page 833 in the textbook). The gradient vector contains important information about how the function f changes. The points that should be emphasized are:

- The length of $\nabla f(a, b)$ is the maximum rate of increase of f at (a, b) . This means that if you are standing on the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ and you start walking in any direction along the surface, the rate of change in height may be different along different directions, but no more than $|\nabla f(a, b)|$. Similarly, the maximum rate of decrease is $-|\nabla f(a, b)|$.
- The maximum rate of increase (respectively decrease) of f at (a, b) takes place along the direction of $\nabla f(a, b)$ (respectively $-\nabla f(a, b)$). For this reason, the length and direction of ∇f are also called the rate and direction of steepest ascent respectively.
- There is no instantaneous change in the height of the function along any direction perpendicular to $\nabla f(a, b)$.

The above information is summarized in Theorem 12.11 on page 835 in the textbook.

In addition to the interpretations above, the components of the gradient vector can be used to obtain approximations to the change in f for simultaneous changes in the values of x and y . For instance, if Δz is the change in the value of f as the independent variables (x, y) change from (a, b) to $(a + \Delta x, b + \Delta y)$, then

$$\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

This formula is known as the *linear approximation* formula (see page 847 of the textbook).

The following items are omitted:

1. directional derivatives (12.6),
2. relation between gradient and level curves (12.6),
3. gradient and directional derivative in three dimensions (12.6),
4. tangent planes (12.7).

1.2. **Section 12.8.** The second half of the week will be spent on local maximum/minimum problems. The content of pages 853–858 will be treated in-depth, covering

- (a) critical points,
- (b) saddle points, and
- (c) the second derivative test.

Problems involving absolute maximum and minimum values on closed bounded sets will be omitted.

2. LEARNING OBJECTIVES

By the end of the week and after going through the practice problems,

1. you should be familiar with the notion of the gradient vector of two-variable functions,
2. you should be able to compute the rate of steepest ascent/descent, direction of maximum increase/decrease and no change,
3. you should be able to read off changes in values of f under changes in values of x and y using the linear approximation formula,
4. you should be able to find the critical points of a function $f(x, y)$, and identify a critical point as a local maximum or minimum or saddle point using the second derivative test.