NAME: ______________________

MATH 105: MIDTERM #1 PRACTICE PROBLEMS

1. TRUE or FALSE, **plus explanation**. Give a full-word answer TRUE or FALSE. If the statement is true, explain why, using concepts and results from class to justify your answer. If the statement is false, give a counterexample.

   (a) [4 points] Suppose the graph of a function f has the following properties: The trace in the plane $z = c$ is empty when $c < 0$, is a single point when $c = 0$, and is a circle when $c > 0$. Then the graph of f is a cone that opens upward.

   Solution: FALSE. We could have $f(x, y) = x^2 + y^2$ (a paraboloid).

   (b) [4 points] If $f(x, y)$ is any function of two variables, then no two level curves of f can intersect.

   Solution: TRUE. Suppose the two level curves are $f(x, y) = c_1$ and $f(x, y) = c_2$, where $c_1 \neq c_2$. If (a, b) is on the first curve, then $f(a, b) = c_1$. But then $f(a, b) \neq c_2$, so (a, b) is not on the second curve. [Remember that part of the definition of a function is that $f(x, y)$ has a single value at each point in the domain.]

   (c) [4 points] Suppose P_1, P_2, and P_3 are three planes in $\mathbf{R}^3$. If P_1 and P_2 are both orthogonal to P_3, then P_1 and P_2 are parallel to each other.

   Solution: FALSE. The planes $x + y = 1$ and $x + 2y = 1$ are both perpendicular to the plane $z = 0$ but are not parallel to each other.

(d) [4 points] If $f(x, y)$ has continuous partial derivatives of all orders, then $f_{xxy} = f_{yxx}$ at every point in $\mathbf{R}^2$.

Solution: TRUE. Applying Clairaut's theorem twice (first to f_x and then to f), we see that $f_{xxy} = f_{xyx} = f_{yxx}$.

(e) [4 points] Suppose that f is defined and differentiable on all of $\mathbf{R}^2$. If there are no critical points of f, then f does not have a global maximum on $\mathbf{R}^2$.

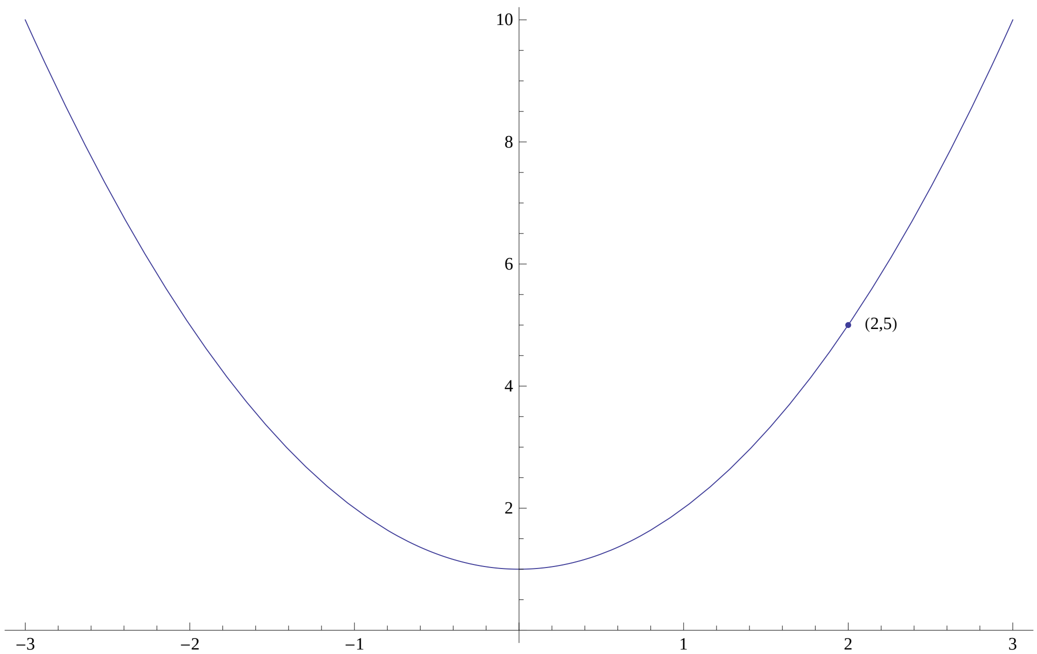
Solution: TRUE. Every global maximum of f is a local maximum. At any local maximum, f has a critical point. So if there are no critical points, then there is no global max.

2. [5 points] Consider the function $f(x, y) = e^{y-x^2-1}$. Find the equation of the level curve of f that passes through the point $(2, 5)$. Then sketch this curve, clearly labeling the point $(2, 5)$.

Solution: We have $f(2, 5) = e^{5-2^2-1} = e^0 = 1$. So the level curve containing $(5, 2)$ is the curve $f(x, y) = 1$.

Applying ln to both sides of the equation $e^{y-x^2-1} = 1$, we can rewrite the equation of the level curve in the form $y - x^2 - 1 = 0$, or $y = x^2 + 1$.

The graph looks like this:



3. Let $f(x, y) = (1 - 2y)(x^2 - xy)$.

(a) [5 points] Compute the partial derivatives f_x and f_y.

Solution: We have $f_x = (1 - 2y)(2x - y)$. Using the product rule, we find that

$$\begin{aligned} f_y &= (1 - 2y)(-x) + (x^2 - xy)(-2) \\ &= (1 - 2y)(-x) + x(x - y)(-2) \\ &= x((2y - 1) + 2(y - x)) \\ &= x(4y - 2x - 1). \end{aligned}$$

(b) [5 points] Using your answer to (a), find all the critical points of f.

Solution: If $x = 0$, then $f_y = 0$ and $f_x = (1 - 2y)(-y)$. So the critical points with $x = 0$ are $(0, 0)$ and $(0, \frac{1}{2})$.

If $x \neq 0$, then $4y - 2x - 1 = 0$, so $2x = 4y - 1$. Substituting this back into the equation $f_x = 0$, we find that $(1 - 2y)(3y - 1) = 0$, so $y = \frac{1}{2}$ or $y = \frac{1}{3}$. In these cases, $2x = 4 \cdot \frac{1}{2} - 1 = 1$ and $2x = 4\frac{1}{3} - 1 = \frac{1}{3}$ (respectively), so $x = \frac{1}{2}$ or $x = \frac{1}{6}$. So the critical points in this case are $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{6}, \frac{1}{3})$.

Combining these two cases, we find that the critical points are $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{6}, \frac{1}{3})$.

(c) [5 points] Apply the second derivative test to label each of the points found in (b) as a `local minimum`, `local maximum`, `saddle point`, or `inconclusive`.

Solution: We have $f_{xx} = 2(1-2y)$ and $f_{yy} = 4x$. Also,

$$\begin{aligned} f_{xy} &= (1-2y)(-1) + (2x-y)(-2) \\ &= 2y - 1 + 2y - 4x \\ &= 4y - 4x - 1. \end{aligned}$$

So at $(0,0)$, we have $f_{xx} = 2$, $f_{yy} = 0$, and $f_{xy} = -1$ so

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -1,$$

so $(0,0)$ is a `saddle point`. At $(0,1/2)$, we have $f_{xx} = 0$, $f_{yy} = 0$, and $f_{xy} = 1$, and

$$D(0,1/2) = f_{xx}(0,1/2)f_{yy}(0,1/2) - f_{xy}(0,1/2)^2 = -1.$$

So $(0,1/2)$ is a `saddle point`. At $(1/2,1/2)$, we have $f_{xx} = 0$, $f_{yy} = 2$, $f_{xy} = -1$, so

$$D(1/2,1/2) = f_{xx}(1/2,1/2)f_{yy}(1/2,1/2) - f_{xy}(1/2,1/2)^2 = -1,$$

so $(1/2,1/2)$ is a `saddle point`. At $(1/6,1/3)$, we have $f_{xx} = 2/3$, $f_{yy} = 2/3$, and $f_{xy} = -1/3$. So

$$\begin{aligned} D(1/6,1/3) &= f_{xx}(1/6,1/3)f_{yy}(1/6,1/3) - f_{xy}(1/6,1/3)^2 \\ &= (2/3)(2/3) - (-1/3)^2 = 1/3. \end{aligned}$$

Since $f_{xx}(1/6,1/3) > 0$, the point $(1/6,1/3)$ is a `local minimum`.

4. [15 points] Find the point (x, y, z) on the plane $x - 2y + 2z = 3$ that is closest to the origin. Show your work and explain your steps.

Solution: The square of the distance from (x, y, z) to the origin is $x^2 + y^2 + z^2$. Since $x - 2y + 2z = 3$, we have $x = 3 + 2y - 2z$, and so our squared distance function is given by

$$D = (3 + 2y - 2z)^2 + y^2 + z^2.$$

We want to find the absolute minimum of $D(x, y)$. Since the absolute minimum is also a local minimum, we can use the method of critical points: Notice that

$$D_y = 4(3 + 2y - 2z) + 2y, \quad D_z = -4(3 + 2y - 2z) + 2z.$$

Setting $D_y = 0$ and $D_z = 0$ gives the system of equations

$$\begin{aligned} 2y &= -4(3 + 2y - 2z) \\ 2z &= 4(3 + 2y - 2z). \end{aligned}$$

Comparing the two equations, we see that $y = -z$. Substituting this into the first equation gives $2(-z) = -4(3 - 2z - 2z) = -4(3 - 4z) = -12 + 16z$. Rearranging, $18z = 12$, so $z = 12/18 = 2/3$. Since $y = -z$, we have $y = -2/3$. Since $x - 2y + 2z = 3$, we have

$$x = 3 + 2y - 2z = \frac{1}{3}.$$

So the minimizing point is $(1/3, -2/3, 2/3)$.

5. (a) [5 points] In your own words, explain what it means for a function $f(x, y)$ to have a `saddle point` at (a, b).

Solution: A saddle point (a, b) is a critical point of f with the following property: In every small disk about (a, b), there is a point (x, y) where $f(x, y) > f(a, b)$ and also a point where $f(x, y) < f(a, b)$.

(b) [5 points] The function $f(x, y) = x^5 - x^2y^3 + y^7 + 11$ has a critical point at $(0, 0)$. (You may assume this without checking it.) Show that $(0, 0)$ is a saddle point of f.

Solution: Notice that $f(0, 0) = 11$. We are given that $(0, 0)$ is a critical point. To show that $(0, 0)$ is a saddle point, we have to show that in every disk about $(0, 0)$, there are points (x, y) with $f(x, y) > 11$ and $f(x, y) < 11$.

To see this, notice that $f(0, y) = y^7 + 11$. So $f(0, y) > 11$ if $y > 0$ and $f(0, y) < 11$ if $y < 0$. Since we can take y as small as we want, $(0, 0)$ is indeed a saddle point.

[You could also try using the second-derivative test here. However, that test would be inconclusive in this case, even though $(0, 0)$ is a saddle point.]

6. [10 points] Find the absolute maximum value of the function $f(x, y) = xy^2$ on the region R consisting of those points (x, y) with $x^2 + y^2 \leq 4$ and $x \geq 0, y \geq 0$. (So R is the portion of the disk of radius 2 centered at the origin which belongs to the first quadrant, boundary points included.) Show your work and explain which methods from class you use.

Solution: We use the method from class for finding global extrema on closed bounded sets.

We first look for critical points of f inside R. We have

$$\frac{\partial f}{\partial x}(x, y) = y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 2xy.$$

The first of these equations shows that at any critical point belonging to R, we have $y = 0$. But then (x, y) is on the boundary of R. So there are no critical points interior to R.

We turn next to the boundary of R. If $x = 0$ or $y = 0$, then $f(x, y) = 0$. Otherwise, (x, y) belongs to the the circular portion of the boundary, and so $x^2 + y^2 = 4$. In that case,

$$f(x, y) = xy^2 = x(4 - x^2) = 4x - x^3.$$

So we have to maximize the function $F(x) = 4x - x^3$ for $0 \leq x \leq 2$. We have $F'(x) = 4 - 3x^2$, so the only critical point of F with $0 \leq x \leq 2$ is $x = \sqrt{4/3}$, where

$$F(x) = x(4 - x^2) = \sqrt{4/3}(4 - 4/3) = \frac{2}{\sqrt{3}} \frac{8}{3} = \frac{16}{3\sqrt{3}}.$$

We also have to check the endpoints of the interval $[0, 2]$: We have $F(0) = 0$ and $F(2) = 0$.

Since the absolute maximum of F is the largest value of F seen so far, we see that this maximum value is $\frac{16}{3\sqrt{3}}$.

7. [10 points] A firm makes x units of product A and y units of product B and has a production possibilities curve given by the equation $x^2 + 25y^2 = 25000$ for $x \geq 0, y \geq 0$. Suppose profits are \$3 per unit for product A and \$5 per unit for product B. Find the production schedule (i.e. the values of x and y) that maximizes the total profit.

Solution: We have to maximize $f(x, y) = 3x + 5y$ subject to the constraint that $g(x, y) = 0$, where $g(x, y) = x^2 + 25y^2 - 25000$. We use Lagrange multipliers and so look for solutions to

$$\nabla f = \lambda \nabla g.$$

Since $\nabla f = \langle 3, 5\rangle$ and $\nabla g = \langle 2x, 50y\rangle$, this equation of vectors becomes the two simultaneous equations

$$\begin{aligned} 3 &= \lambda(2x) \\ 5 &= \lambda(50y). \end{aligned}$$

Dividing one equation by the other, we find that

$$\frac{5}{3} = \frac{50y}{2x} = \frac{25y}{x},$$

so that (cross-multiplying)

$$5x = 75y, \quad \text{or} \quad x = 15y.$$

Since $x^2 + 25y^2 = 25000$, this gives that $250y^2 = 25000$, or (dividing by 250) that $y^2 = 100$, so that $y = 10$ (since $y \geq 0$). Since $x = 15y$, we have $x = 150$. So the optimal production schedule is $x = 150$ and $y = 10$.

8. In this problem, we guide you through the computation of the area underneath one hump of the curve $y = \sin x$.

   (a) [5 points] Write down the `right-endpoint Riemann sum` for the area under the graph of $y = \sin x$ from $x = 0$ to $x = \pi$, using n subintervals.

Solution: Here $[a, b] = [0, \pi]$, the length of each subinterval is $\Delta x = \frac{\pi}{n}$, and each $x_k = 0 + k\Delta = k\frac{\pi}{n}$. So the Riemann sum is

$$\sin\left(\frac{\pi}{n}\right)\frac{\pi}{n} + \sin\left(2\frac{\pi}{n}\right)\frac{\pi}{n} + \cdots + \sin\left(n\frac{\pi}{n}\right)\frac{\pi}{n},$$

or, in Σ-notation,

$$\sum_{k=1}^{n} \sin\left(k\frac{\pi}{n}\right)\frac{\pi}{n}.$$

   (b) [10 points] Now assume the validity of the following formula (for each real number θ):

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \sin(\frac{1}{2}(n+1)\theta).$$

Using this formula, compute the limit as $n \to \infty$ of the expression found in part (a) and thus evaluate the area exactly. [*Hint:* The identity $\sin(\frac{\pi}{2} + \frac{\pi}{2n}) = \cos(\frac{\pi}{2n})$ may be useful.]

Solution: Using the formula and the hint,

$$\begin{aligned}\sum_{k=1}^{n} \sin\left(k\frac{\pi}{n}\right)\frac{\pi}{n} &= \frac{\pi}{n}\frac{\sin(\pi/2)}{\sin(\frac{\pi}{2n})} \sin(\frac{\pi}{2} + \frac{\pi}{2n}) \\ &= 2\cos\left(\frac{\pi}{2n}\right)\frac{\pi/(2n)}{\sin(\pi/(2n))}.\end{aligned}$$

Taking the limit as $n \to \infty$ gives

$$2\left(\lim_{n\to\infty} \cos(\pi/2n)\right)\left(\lim_{n\to\infty} \frac{\pi/(2n)}{\sin(\pi/2n)}\right),$$

as long as both right-hand limits exist. Since $\frac{\pi}{2n} \to 0$ as $n \to \infty$, the first limit is $\cos(0) = 1$, and the second limit is

$$\lim_{\theta\to 0} \frac{\theta}{\sin(\theta)} = 1.$$

(You should remember this last result from differential calculus, when you discussed derivatives of trig functions.) Thus,

$$\int_0^{\pi} \sin(x)\,dx = 2 \cdot 1 \cdot 1 = 2.$$