Antiderivatives or indefinite integrals

Definition

If F and f are two functions on $[a, b]$ such that

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For instance, we know that

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}x^3 = 3x^2.$$

We say that

- $\cos x$ and $3x^2$ are **the** derivatives of $\sin x$ and x^3 respectively, or
- $\sin x$ is **an** antiderivative of $\cos x$,
- x^3 is **an** antiderivative of $3x^2$.

Remarks on antiderivatives

- Given F, its derivative f is unique.
- However given f, its antiderivative F is not unique, but determined upto an additive constant. In other words, if F is an antiderivative of f, so is $F + C$ for any arbitrary constant C, since

$$\frac{d}{dx}(F(x) + C) = F'(x) + \frac{d}{dx}C = F'(x) + 0 = f(x).$$

- The antiderivative of f is denoted by an **indefinite integral** (an elongated S same as a definite integral, but without the upper and lower endpoints):

$$\int f(x)\,dx = F(x) + C.$$

Examples of indefinite integrals : the power rule

The derivative identities

$$\frac{d}{dx}\left[\frac{x^{a+1}}{a+1}\right] = x^a \text{ for } a \neq -1,$$
$$\frac{d}{dx}\ln x = \frac{1}{x},$$

may be rephrased as

$$\int x^a\,dx = \left\{ \begin{array}{c} \dfrac{x^{a+1}}{a+1} + C \text{ if } a \neq -1, \\ \ln x + C \text{ if } a = -1. \end{array} \right\}$$

Indefinite integrals: some more examples

Exponential and trigonometric functions

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C,$$
$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C,$$
$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C,$$
$$\int sec^2 ax\,dx = \frac{1}{a}\tan(ax) + C,$$
$$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + C.$$

An exercise

Compute the indefinite integral

$$\int \frac{x - \sqrt{x}}{x^3}\, dx.$$

A. $-\frac{1}{x} - \frac{2x^{-7/2}}{7} + C$

B. $-\frac{1}{x} + \frac{2x^{-7/2}}{7} + C$

C. $-\frac{1}{x}(1 + \frac{2}{3\sqrt{x}}) + C$

D. $-\frac{1}{x}(1 - \frac{2}{3\sqrt{x}}) + C$

Definite vs indefinite integral: is there a connection?

The Area function

Given a continuous function f on $[a, b]$, define its area function $A(x)$ as the definite integral

$$A(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b.$$

In other words, $A(x)$ is a function on $[a, b]$ whose value at x is the signed area under the curve $y = f(t)$ between $t = a$ and $t = x$.

Computation of the area function: Example 1

Given a constant function f, its area function is

A. constant

B. linear

C. quadratic

D. always positive

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Question: What is the derivative of the area function of a constant function?

Area function: Example 2

For a linear function $f(t) = mt + c$, its area function on $[0, b]$ is

A. constant

B. linear

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D. always increasing

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For a linear function $f(t) = mt + c$, its area function on $[0, b]$ is

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B. linear

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D. always increasing

Question: What is the derivative of the area function of a linear function?