Homework 10 - Math 321, Spring 2012

Due on Friday March 30

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1. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a fixed partition of $[a, b]$, and let α be an increasing step function on $[a, b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has (possibly) jumps of size α_i at each of the point x_i, where

$$\alpha_i = \alpha(x_i+) - \alpha(x_i-) \text{ for } 0 < i < n, \quad \alpha_0 = \alpha(a+) - \alpha(a), \quad \alpha_n = \alpha(b) - \alpha(b-).$$

   If f is continuous at each of the points x_i (with appropriate one-sided analogues at a and b) show that $f \in \mathcal{R}_\alpha$, and

$$\int_a^b f\, d\alpha = \sum_{i=0}^{n} f(x_i)\alpha_i.$$

2. Let $\alpha \in \mathrm{BV}[a, b]$ be right continuous. Given $\epsilon > 0$ and a partition P of $[a, b]$, construct $f \in C[a, b]$ with $||f||_\infty \leq 1$ such that

$$\int_a^b f\, d\alpha \geq V(\alpha, P) - \epsilon.$$

   Conclude that

$$V_a^b \alpha = \sup \left\{ \int_a^b f\, d\alpha : ||f||_\infty \leq 1 \right\}.$$

3. A few weeks ago (HW 7, problem 3) you proved a result called Helly's first theorem: any bounded sequence in $\mathrm{BV}[a, b]$ has a pointwise convergent subsequence. Now prove its companion, called Helly's second theorem.

   Suppose that α_n is a sequence in $\mathrm{BV}[a, b]$. If $\alpha_n \to \alpha$ pointwise on $[a, b]$, and if $V_a^b \alpha_n \leq K$ for all n, then $\alpha \in \mathrm{BV}[a, b]$, and

$$\int_a^b f\, d\alpha_n \to \int_a^b f\, d\alpha \quad \text{for all } f \in C[a, b].$$

4. (a) Given $\alpha \in \mathrm{BV}[a, b]$, define

$$\beta(a) = \alpha(a), \quad \beta(x) = \alpha(x+), \quad \beta(b) = \alpha(b).$$

   Show that β is right continuous on (a, b), that $\beta \in \mathrm{BV}[a, b]$, and that

$$\int_a^b f\, d\alpha = \int_a^b f\, d\beta \quad \text{for every } f \in C[a, b].$$

   (b) Given $\alpha \in \mathrm{BV}[a, b]$, show that there is a unique $\beta \in \mathrm{BV}[a, b]$ with $\beta(a) = 0$ such that β is right continuous on (a, b) and

$$\int_a^b f\, d\alpha = \int_a^b f\, d\beta \quad \text{for every } f \in C[a, b].$$