Solutions to MATH 300 Homework 8

EXERCISES 5.3

3. (e) Using the results of Prob. 2, the radius of convergence of the power series is $R = 1/L$, where

$$L = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(3-i)^{k+1}}{(k+1)^2}}{\frac{(3-i)^k}{k^2}}\right| = |3-i|\lim_{k\to\infty}\frac{k^2}{(k+1)^2} = \sqrt{10}.$$

So the circle of convergence is $|z+2| = 1/\sqrt{10}$.

5. (b) In the circle $\{z \in \mathbb{C} : |z| = 1\}$,

$$\begin{aligned}\oint_{|z|=1}\frac{f(z)}{z^4}\,dz &= \oint_{|z|=1}\Big(\frac{1}{3z^3}+\frac{8}{9z^2}+\frac{1}{z}+\frac{64}{81}+\frac{125}{243}z+\cdots+\frac{k^3}{3^k}z^{k-4}+\cdots\Big)\,dz\\ &= 0+0+2\pi i+0+\cdots+0+\cdots\\ &= 2\pi i.\end{aligned}$$

6. (a) We know from **Sec. 5.2 Example 2** that the Maclaurin expansion for $\sin z$ is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.$$

Then

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

for all $z \in \mathbb{C}$.

(b) Clearly, $f(z)$ has derivative for any $z \neq 0$. At $z = 0$,

$$f'(0) = \lim_{z\to 0}\Big(\frac{\sin z}{z} - 1\Big) = \lim_{z\to 0}\Big(\frac{\sin z - z}{z}\Big) = \lim_{z\to 0}(\cos z - 1) = 0,$$

where the 3rd equality results from Maclaurin expansion of $\sin z$ and $\cos z$ at $z = 0$. Therefore, f has derivative in a neighbourhood of the origin, hence is analytic there.
(c) $f^{(3)}(0) = 3!a_3 = 0$.
(d) $f^{(4)}(0) = 4!a_4 = 4!(1/5!) = 1/5$.

10. Applying Prob. 2, $\lim_{k\to\infty} |\frac{a_k}{a_{k+1}}| = R = \lim_{k\to\infty} |\frac{ka_k}{(k+1)a_{k+1}}|$.

EXERCISES 5.5

2. (b) No. Recall that the principal branch of $\sqrt{z}$ is analytic in the slit domain $D^* = \mathbb{C}\backslash(-\infty, 0]$. Thus, the condition of **Theorem 14** does not satisfy.

3. (b)

$$f(z) = \frac{1}{3z}\left[\frac{1}{1-(-\frac{1}{z})}\right] - \frac{1}{3}\left[\frac{1}{1-\frac{z}{2}}\right] = \frac{1}{3}\sum_{j=1}^{\infty}(-1)^{j-1}z^{-j} - \frac{1}{3}\sum_{j=0}^{\infty}(\frac{1}{2})^j z^j.$$

6. As we already know,

$$\cos w = \sum_{j=0}^{\infty}(-1)^j \frac{w^{2j}}{(2j)!},$$

for all (finite) w. Therefore, the Laurent series for $z^2\cos(\frac{1}{3z})$ is

$$\sum_{j=0}^{\infty}(-1)^j \frac{z^{2-2j}}{3^{2j}(2j)!}.$$

7. (b) As we know,

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}.$$

Then

$$\frac{1}{e^z - 1} = \frac{1}{\sum_{k=1}^{\infty}\frac{z^k}{k!}} = \frac{1}{z}\frac{1}{\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!}} := \frac{1}{z}\sum_{j=0}^{\infty} a_j z^j,$$

where

$$(\sum_{j=0}^{\infty} a_j z^j)\left(\frac{1}{\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!}}\right) = 1.$$

Therefore, comparing the coefficients gives us

$$a_0 = 1,\ a_1 + \frac{1}{2}a_0 = 0,\ a_2 + \frac{1}{2}a_1 + \frac{1}{6}a_0 = 0, \cdots$$

Then

$$a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{12}, \cdots$$

9. Consider the cases $j \geqslant 0$ and $j < 0$. Then the series can be written as

$$\sum_{j=-\infty}^{\infty} \frac{z^j}{2^{|j|}} = \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j + \sum_{j=1}^{\infty} \frac{1}{(2z)^j}.$$

Then in order that these geometric series converge, we need $\left|\frac{1}{2z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$, whence $\frac{1}{2} < |z| < 2$.