Math 300 Midterm I Solutions

1. **Find the modulus and argument for each of the complex numbers below. Give the unique value of the argument that lies in the interval $[0, 2\pi)$.**

   (a) $\dfrac{2}{i} + \dfrac{i}{5}$.

   Solution. Note that
   $$z = \frac{2}{i} + \frac{i}{5} = \left(-2 + \frac{1}{5}\right) i = -\frac{9}{5} i.$$
   Hence, $|z| = \frac{9}{5}$ and $\operatorname{Arg}(z) = \frac{3\pi}{2}$. □

   (b) $\left(\dfrac{1+i\sqrt{3}}{2}\right)^{3600}$.

   Solution. Note that
   $$w = \left(\frac{1+i\sqrt{3}}{2}\right)^{3600} = \left(e^{\pi i/3}\right)^{3600} = e^{1200\pi i} = 1.$$
   Hence, $|w| = 1$ and $\operatorname{Arg}(w) = 0$. □

2. **Find all solutions of the equation**
   $$z^4 = 8iz,$$
   and express them in the form $a + ib$ where a and b are real numbers.

   Solution. $z = 0$ or $z^3 = 8i = 8e^{\pi i/2} = 8e^{5\pi i/2} = 8e^{9\pi i/2}$.
   Hence, the solutions are
   $$z = 0,$$
   $$z = 2e^{\pi i/6} = \sqrt{3} + i,$$
   $$z = 2e^{5\pi i/6} = -\sqrt{3} + i,$$
   $$z = 2e^{9\pi i/6} = -2i.$$
   □

3. **Let $v(x, y) = 5x - xy + 4$.**

   (a) **Show that $v(x, y)$ is harmonic in the entire plane.**

   Solution. $v_x = 5 - y$, $v_y = -x$, $v_{xx} = 0$, $v_{yy} = 0$.
   Hence $v_{xx} + v_{yy} = 0 + 0 = 0$, implying $v(x, y)$ is harmonic in the entire plane. □

(b) **Construct an entire function $f(z)$ such that $\mathbf{Im}\{f(z)\} = v(x, y)$.**

Solution. Let $f = u + iv$.
Then, $u_x = v_y = -x$, so $u = \int -x = -\frac{x^2}{2} + \phi(y)$.
From $u_y = -v_x = y - 5$, we have $\phi'(y) = y - 5$, and so $\phi(y) = \frac{y^2}{2} - 5y + c$, where $c \in \mathbb{R}$ is any constant.
Choosing $c = 0$, we get $f = -\frac{x^2}{2} + \frac{y^2}{2} - 5y + i(5x - xy + 4)$. □

4. (a) **Show that if $f(z)$ and $\overline{f(z)}$ are analytic in a domain D, then $f(z)$ is constant in D.**

Solution. Suppose that $f = u = iv$ and $\overline{f} = u - iv$ are both analytic on a domain D. Then both the pairs (u, v) and $(u, -v)$ obey the Cauchy-Riemann equations:

$$\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases} \qquad \begin{cases} u_x &= -v_y \\ u_y &= v_x. \end{cases}$$

Combining the equations above, we obtain that $u_x = u_y = 0$ and $v_x = v_y = 0$. Thus u and v are both constant functions, hence f is constant on D. □

(b) **Using part (a), show that $p(\overline{z})$ is not analytic in any domain of the complex plane if p is a polynomial with degree at least 1.**

Solution. We argue by contradiction. Let p be a polynomial of degree at least n; i.e., $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $n \geq 1$ and the coefficients a_j are complex numbers. In particular, there exists at least one $j \geq 1$ such that $a_j \neq 0$.
Suppose if possible that $f(z) = p(\overline{z}) = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \cdots + a_n \overline{z}^n$ is analytic. On the other hand, we observe that $\overline{f(z)} = \overline{a}_0 + \overline{a}_1 z + \cdots + \overline{a}_n z^n$ is a polynomial of degree at least 1 (since $a_j \neq 0$ implies $\overline{a}_j \neq 0$). A polynomial function is known to be analytic on all of $\mathbb{C}$. Thus $f(z)$ and $\overline{f(z)}$ are both analytic. By the result of part (a), $f(z)$ must be a constant function. This means that the polynomial $\overline{f(z)}$ is a constant function, which contradicts the fact that it is assumed to be of degree 1. □

5. Find the partial fraction decomposition of

$$R(z) = \frac{2}{z(1-z)^2}.$$

Solution. Let

$$\frac{2}{z(1-z)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2}.$$

Then,

$$2 = A(z-1)^2 + Bz(z-1) + Cz.$$

Putting $z = 0$, we get $A = 2$.
Putting $z = 1$, we get $C = 2$.

Comparing coefficient of z^2, we get $A + B = 0$ and so $B = -2$.
Thus,

$$\frac{2}{z(1-z)^2} = \frac{2}{z} - \frac{2}{z-1} + \frac{2}{(z-1)^2}.$$

□

6. For each of the statements below, indicate whether they are true or false. **If true, give a proof. If false, give a counter example.**

(a) $|e^{-z}| \leq 1$ if $|z| \leq 1$.

Solution. The statement is false. The number $z = -1$ obeys $|z| \leq 1$, but $|e^{-z}| = e > 1$. □

(b) $\mathrm{Arg}(\mathrm{Re}(z)) = 0$ for any complex number z. Here "Arg" denotes the value of the argument that lies in the interval $(-\pi, \pi]$.

Solution. The statement is false. Nonzero real numbers have argument 0 if they are positive, and π if they are negative. For any complex number z with a negative real part x, say $z = -1 + i$, $\mathrm{Arg}(\mathrm{Re}(z)) = \mathrm{Arg}(x) = \pi$. □

(c) The equation $e^z = -1$ has no solution in $\mathbb{C}$.

Solution. The statement is false. $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$, so $z = i\pi$ is a solution. □

(d) The function $f(z) = \frac{\bar{z}-1}{|z|^2 - z}$ is rational.

Solution. Since $|z|^2 = z\bar{z}$, the function f can be simplified as $f(z) = \frac{\bar{z}-1}{z(\bar{z}-1)} = \frac{1}{z}$ if $z \neq 1$. The function $1/z$ is rational, being the ratio of two polynomials (the constant function 1 and the function z. □

(e) $-z^4 - 1 < 0$ for all $z \in \mathbb{C}$.

Solution. The statement is false. If we choose z to be a square root of i, say $z = e^{i\frac{\pi}{4}}$, then $z^4 = i^2 = -1$, so $-z^4 - 1 = 1 - 1 = 0$. □