[3] 1. (a) Express $\dfrac{(3-i)^2}{4+2i}$ in the form $x+iy$, where x and y are real numbers.

$$\frac{(3-i)^2}{4+2i} = \frac{8-6i}{4+2i}$$
$$= \frac{4-3i}{2+i}$$
$$= \frac{(4-3i)(2-i)}{5}$$
$$= \frac{5-10i}{5}$$
$$= 1-2i$$

[4] (b) Find all complex numbers that satisfy $z = (-8)^{\frac{1}{3}}$. Write your answer in the form $z = x+iy$ where x and y are real numbers.

$$(-8)^{\frac{1}{3}} = 2e^{\frac{\pi i}{3} + \frac{2k\pi i}{3}} = 2\left(\cos\frac{\pi+2k\pi}{3} + i\sin\frac{\pi+2k\pi}{3}\right)$$

$k=0,\quad 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1+\sqrt{3}i$

$k=1,\quad 2\left(\cos\pi + i\sin\pi\right) = -2$

$k=2,\quad 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1-\sqrt{3}i$

[3] (c) Let $z = (1+i)^{2014}$. Find $|z|$ and $\operatorname{Arg}(z)$.

$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$
$$(1+i)^{2014} = 2^{1007} e^{i\left(-\frac{\pi}{2}\right)}$$
$$\left|(1+i)^{2014}\right| = 2^{1007}$$
$$\operatorname{Arg} z = -\frac{\pi}{2}$$

2. Find all values of z in $\mathbb{C}$ where $f(z) = x^3 + iy^3$ is

(a) differentiable

Solution: In order for f to be differentiable at any z, the Cauchy-Riemann equations must be true at z. For this problem, $u = x^3$ and $v = y^3$, so

$$\begin{aligned} u_x &= 3x^2 \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 3y^2. \end{aligned}$$

The Cauchy-Riemann equations then say that $u_x = v_y$, or $3x^2 = 3y^2$, and $u_y = -v_x$, or $0 = -0$. The Cauchy-Riemann equations are then satisfied exactly when $3x^2 = 3y^2$, which happens when $x^2 = y^2$ or $x = \pm y$. So, these are the only values of z for which f could possibly be differentiable. To know that f is actually differentiable, we also must check that the partial derivatives u_x, u_y, v_x, and v_y are all continuous at these points. But these partials are polynomials and so continuous everywhere. Therefore, f is differentiable precisely at those values of z for which $x = \pm y$.

(b) analytic

Solution: In order for f to be analytic at a point z, there must be a disk centered at z in which f is differentiable at every point. Since the set of points where f is differentiable is just a union of two lines, there is no disk at all in which f is differentiable at every point, and so f is not analytic at any point of $\mathbb{C}$.

3. Let $v(x, y) = y^3 - 3x^2y + 4xy - x$.

[3] (a) Show that v is harmonic.

$v_x = -6xy + 4y - 1, \quad v_{xx} = -6y$

$v_y = 3y^2 - 3x^2 + 4x, \quad v_{yy} = 6y$

$\left.\begin{matrix} \\ \end{matrix}\right\} v_{xx} + v_{yy} = 0.$

$v_{xx}, v_{yy}, v_{xy}, v_{yx}$ are continuous

$\therefore v$ harmonic.

[7] (b) Find a function $f(z)$ that is analytic on the entire complex plane such that $v(x, y)$ is the imaginary part of f and such that $f(1 + i) = 2 + i$. You may leave your answer as a function in x and y, rather than a function of z.

$f = u + iv$

$u_x = v_y = 3y^2 - 3x^2 + 4x \quad (1)$

$u_y = -v_x = 6xy - 4y + 1 \quad (2)$

Integrating (1) in x:

$u(x,y) = 3y^2x - x^3 + 2x^2 + h(y) \quad (3)$. Differentiating (3) in y:

$u_y = 6xy + h' \quad (4) \qquad \therefore h'(y) = -4y + 1$ by (2), (4)

$\therefore h(y) = -2y^2 + y + C$

$\therefore f(z) = (3xy^2 - x^3 + 2x^2 - 2y^2 + y + C) + i(y^3 - 3x^2y + 4xy - x)$

$2 + i = (3 - \cancel{1} \cancel{+2} \cancel{-2} + 1 + C) + i(1 - 3 + 4 - 1)$

$= (3 + C) + i \qquad \therefore C = -1.$

CR eqns hold everywhere on $\mathbb{C}$ and u_x, u_y, v_x, v_y are continuous there. So f is analytic on $\mathbb{C}$.

In fact, $f(z) = -z^3 + 2z^2 - iz - 1$.

[10] **4.** Find all complex numbers z that satisfy

$$\sin z + \cos z = 1.$$

Write your answers in the form $z = x + iy$ where x and y are real numbers.

$$\frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2} = 1.$$

set $w = e^{iz}$

$$\frac{w - \frac{1}{w}}{2i} + \frac{w + \frac{1}{w}}{2} = 1$$

$$\xRightarrow{\text{multiply } 2i} \quad w - \frac{1}{w} + i\left(w + \frac{1}{w}\right) = 2i$$

$$\xRightarrow{\text{multiply } w} \quad w^2 - 1 + iw^2 + i = 2iw$$

$$\therefore \quad (1+i)w^2 - 2iw + (-1+i) = 0$$

$$w = \frac{2i \pm \sqrt{-4 - 4(1+i)(-1+i)}}{2(1+i)}$$

$$= \frac{i \pm \sqrt{-1+2}}{1+i} = \frac{i \pm 1}{1+i} = \begin{cases} 1 \\ \frac{i-1}{1+i} = \frac{(i-1)(1-i)}{2} = \frac{2i}{2} = i \end{cases}$$

Case 1: $w = 1$. $\therefore e^{iz} = 1$

$\therefore iz = 2k\pi i \qquad z = 2k\pi, \quad k \in \mathbb{Z},$

Case 2: $w = i$, $\therefore e^{iz} = i = e^{\frac{\pi}{2}i}$

$\therefore iz = \frac{\pi}{2}i + 2k\pi i \quad \therefore \quad z = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}.$