Marks

[9] **1.** Define

(a) uniform convergence of a sequence of functions

(b) an algebra of functions that vanishes nowhere

(c) an atlas and a maximal atlas

Continued on page 3

[16] **2.** Give an example of each of the following, together with a brief explanation of your example. If an example does not exist, explain why not.

(a) a function $f : [0, 1] \to \mathbb{R}$ which is Riemann integrable on $[0, 1]$ but for which the function $F : [0, 1] \to \mathbb{R}$ defined by $F(x) = \int_0^x f(t)\, dt$ is not Riemann integrable on $[0, 1]$

(b) a sequence of functions that converges to zero pointwise on $[0, 1]$ and uniformly on $[\varepsilon, 1-\varepsilon]$ for every $\varepsilon > 0$, but does not converge uniformly on $[0, 1]$

(c) a Fourier series $\sum\limits_{n=-\infty}^{\infty} c_n e^{inx}$ that does not converge in the mean

(d) two charts for $(-1, 1)$ (with the usual metric) that are not compatible

Continued on page 4

Continued on page 5

[15] **3.** Let $\alpha, f, g : [a, b] \to \mathbb{R}$ with α an increasing function.

(a) Prove that

$$\overline{\int_a^b}(f+g)\,d\alpha \le \overline{\int_a^b} f\,d\alpha \;+ \overline{\int_a^b} g\,d\alpha$$

(b) Either prove that

$$\overline{\int_a^b}(f+g)\,d\alpha = \overline{\int_a^b} f\,d\alpha \;+ \overline{\int_a^b} g\,d\alpha$$

or provide a counterexample.

Continued on page 6

Continued on page 7

[15] **4.** Let $f : [0, 1] \to \mathbb{R}$ have a continuous derivative. Prove *directly from the definition of "integral"* that $\int_0^1 f'(x)\ dx$ exists and equals $f(1) - f(0)$.

Continued on page 8

[15] **5.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a uniformly convergent sequence of continuous real–valued functions defined on a metric space M and let g be a continuous function on $\mathbb{R}$. Define, for each $n \in \mathbb{N}$, $h_n(x) = g\big(f_n(x)\big)$.

(a) Let $M = [0, 1]$. Prove that the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly on $[0, 1]$.

(b) Let $M = \mathbb{R}$. Either prove that the sequence $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$ or provide a counterexample.

Continued on page 9

Continued on page 10

[15] **6.** Let f, f_0, f_1, $\cdots$ be real–valued Riemann integrable functions on the bounded interval $[a, b]$. Assume that

$$\int_a^b f_n(x) f_m(x)\ dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Prove that

$$\lim_{n\to\infty} \int_a^b f(x) f_n(x)\, dx = 0$$

Continued on page 11

Continued on page 12

[15] **7.** Let $\alpha > 0$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Hölder continuous of exponent α if the quantity

$$\|f\|_\alpha = \sup_{x \neq y} \tfrac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is finite. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Hölder continuous real valued functions on $\mathbb{R}$ that obey $\sup_{x\in\mathbb{R}} |f_n(x)| \le 1$ and $\|f_n\|_\alpha \le 1$ for all $n \in \mathbb{N}$. Prove that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ that converges pointwise to f and that furthermore converges uniformly to f on $[-M, M]$ for every $M > 0$.

Continued on page 13

The End

Be sure that this examination has 13 pages including this cover

The University of British Columbia

Sessional Examinations - April 2008

Mathematics 321

Real Variables II

Closed book examination

Time: $2\frac{1}{2}$ hours

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

Special Instructions:

No calculators, notes, or other aids are allowed.

Rules Governing Formal Examinations

1. Each candidate must be prepared to produce, upon request, a UBCcard for identification.

2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

3. No candidate shall be permitted to enter the examination room after the expiration of one half hour from the scheduled starting time, or to leave during the first half hour of the examination.

4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:

(a) having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;

(b) speaking or communicating with other candidates; and

(c) purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		9
2		16
3		15
4		15
5		15
6		15
7		15
Total		100