Homework 8 - Math 321, Spring 2015

Due on Friday March 27

---

1. Helly's first and second theorems were critical components of our proof of the Riesz representation theorem for continuous linear functionals on $\mathcal{C}[a,b]$. You have already proved Helly's first theorem in a previous assignment. Prove the second one here.

   Suppose that $\{\alpha_n\}$ is a sequence in $\mathrm{BV}[a,b]$. If $\alpha_n \to \alpha$ pointwise on $[a,b]$, and if $V_a^b \alpha_n \leq K$ for all n, then $\alpha \in \mathrm{BV}[a,b]$ and
$$\int_a^b f\, d\alpha_n \to \int_a^b f\, d\alpha \quad \text{for all } f \in \mathcal{C}[a,b].$$

2. This problem addresses the issue of uniqueness of the integrator α in Riesz representation theorem. Fill in the following steps.

   (a) Given $\alpha \in \mathrm{BV}[a,b]$, define $\beta(a) = \alpha(a)$, $\beta(x) = \alpha(x+)$ for $a < x < b$ and $\beta(b) = \alpha(b)$. Show that β is right continuous on (a,b), that $\beta \in \mathrm{BV}[a,b]$ and that $\int_a^b f\, d\alpha = \int_a^b f\, d\beta$ for every $f \in \mathcal{C}[a,b]$.

   (b) Given $\alpha \in \mathrm{BV}[a,b]$, show that there is a unique $\beta \in \mathrm{BV}[a,b]$ with $\beta(a) = 0$ such that β is right continuous on (a,b) and $\int_a^b f\, d\alpha = \int_a^b f\, d\beta$ for every $f \in \mathcal{C}[a,b]$.

   (c) Given a continuous linear functional $L : \mathcal{C}[a,b] \to \mathbb{R}$, we constructed a function $\alpha \in \mathrm{BV}[a,b]$ such that
$$L(f) = \int_a^b f\, d\alpha.$$
   Argue that such α is not unique in general. However, combine the above steps to conclude that α can be chosen to be right continuous on (a,b) with $\alpha(a) = 0$, and that in this case, α is unique.

3. For $f \in \mathcal{C}[0,1]$, does the following limit exist? If yes, evaluate it. If not, explain why not.
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} f\left(\frac{\log k}{\log n}\right).$$

4. Many of the results we have studied in this course thus far focus on approximation of continuous functions by other special functions, such as polynomials or trigonometric sums or polygonal functions, usually in the sup norm. Let us now think about approximation in the L^2 norm of Riemann-integrable functions, which are not necessarily continuous, by continuous functions. Suppose that f is a bounded, 2π-periodic function that is Riemann-integrable on $[-\pi, \pi]$.

   (a) Show that there is a continuous function g on $[-\pi, \pi]$ satisfying $||f - g||_2 < \epsilon$.

   (b) Can the function g above be chosen to be continuous and 2π-periodic?

   (c) Can g be chosen to be a trigonometric polynomial?