

Homework 9 - Math 321, Spring 2015

Due on Wednesday April 8

- Let f be a differentiable 2π -periodic function with continuous first derivative. Is f the uniform limit of its partial Fourier sums?
- Determine whether the following statement is true or false: If $f : \mathbb{R} \rightarrow \mathbb{R}$ is 2π -periodic and Riemann-integrable on $[-\pi, \pi]$, then $\|f_\epsilon - f\|_2 \rightarrow 0$ as $\epsilon \rightarrow 0$. Here f_ϵ denotes the translated function $f_\epsilon(x) = f(x + \epsilon)$.
- (a) Obtain the Fourier series of the 2π -periodic function that coincides with $f(t) = (\pi - t)^2$ on $[0, 2\pi]$.
(b) Does f match its Fourier series? Give reasons for your answer.
(c) Use your results from above to derive the identity: $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$.
- (a) Show that the Fourier series of a function f can alternatively be written in the form

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k)e^{ikx}, \quad \text{where} \quad \widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} dt$$

is referred to as the k th Fourier coefficient.

- (b) Determine the relation of $\widehat{f}(k)$ with $\widehat{g}(k)$ in each of the following cases:
 - g is a translate of f , namely $g(x) = f(x + \alpha)$.
 - g is a modulation of f , namely $g(x) = f(x)e^{-i\alpha x}$.
- (c) Given two bounded, 2π -periodic functions f and g , both of which are Riemann-integrable on $[-\pi, \pi]$, define their convolution $h = f * g$ as follows,

$$h(x) = f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t)g(t) dt.$$

Note that the n th partial Fourier sum $s_n f$ and the n th Cesàro sum $\sigma_n f$ are both given in terms of convolutions of f with appropriate kernels. Find \widehat{h} in terms of \widehat{f} and \widehat{g} .

- (d) Use this and the Fourier coefficients of K_n to derive an explicit formula for the n th Cesàro sum $\sigma_n f$ of a Fourier series.
- Let $f \in \mathcal{C}^{2\pi}$, the class of continuous 2π -periodic functions on $[-\pi, \pi]$. We have seen that the partial Fourier sums $s_n(f)$ are excellent approximations of f in L^2 norm, but so far only have indirect evidence that they may not be very good approximations in the sup norm. This problem attempts to make this intuition precise.
 - Show that for every $n \geq 1$, there exists $f_n \in \mathcal{C}^{2\pi}$ such that $\|f_n\|_\infty = 1$ and $\sup_j |s_j f_n(0)| > n$.
 - Use the functions f_n in part (a) to find a single $f \in \mathcal{C}^{2\pi}$ whose Fourier series diverges at 0.

(c) Now modify your construction in part (b) to create a continuous 2π -periodic function whose Fourier series diverges at a dense set of points.

6. This exercise is designed to study a curious property of a certain class of Fourier series, known as Gibbs phenomenon. Discovered by Wilbraham (1849) and studied by Gibbs (1899), this phenomenon refers to the manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a jump discontinuity. The n th partial Fourier sums oscillate near the jump point, which is understandable, but the strange thing is that the oscillation might result in increasing the maximum of the partial sum above that of the function itself. Even more strange is the fact that the overshoot does not die out as you take larger and larger sums (i.e. the frequency increases), but approach a finite nonzero limit! Here is an example where you can see Gibbs phenomenon in action.

Let $f(x) = \text{sgn}(x)$, which takes on the value 1, -1 or 0 according as x is positive, negative or zero.

(a) Show that

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} \quad \text{for every } x \in [-\pi, \pi].$$

(b) Denote by s_n the n th partial sum of the above series. Show that

$$s_n(x) = \frac{2}{\pi} \int_0^x \frac{\sin 2nt}{\sin t} dt.$$

(c) Examine the local maxima and minima of s_n , and deduce that the largest value of s_n is attained at $\frac{\pi}{2n}$.

(d) Interpret $s_n(\frac{\pi}{2n})$ as a Riemann sum and prove that

$$\lim_{n \rightarrow \infty} s_n\left(\frac{\pi}{2n}\right) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt.$$

The value of this limit is about 1.179. Thus, although f has a jump equal to 2 at the origin, the graphs of the approximating curves s_n tend to approximate a vertical segment of length 2.358 in the vicinity of the origin!