Math 121: Homework 1 solutions

1. (a)

$$\begin{aligned}
2^2 - 3^2 + 4^2 - 5^2 + \cdots - 99^2 &= \sum_{i=2}^{99} (-1)^i i^2 \\
&= \sum_{i=1}^{49} [(2i)^2 - (2i+1)^2] = \sum_{i=1}^{49} [4i^2 - 4i^2 - 4i - 1] \\
&= -\sum_{i=1}^{49} [4i+1] = -4\frac{49 \times 50}{2} - 49 \\
&= -4949.
\end{aligned}$$

(b)

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} = \sum_{i=1}^{n} \frac{i}{2^i}.$$

Let $s = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n}$. Then

$$\frac{s}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \cdots + \frac{n}{2^{n+1}}.$$

Subtracting these two sums, we get

$$\begin{aligned}
\frac{s}{2} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} \\
&= \frac{1}{2}\frac{1 - (1/2^n)}{1 - 1/2} - \frac{n}{2^{n+1}} \\
&= 1 - \frac{n+2}{2^{n+1}}.
\end{aligned}$$

Thus,

$$s = 2 - \frac{n+2}{2^n}.$$

2. The formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n = k \geq 1$; that is, $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. Then for $n = k+1$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\
&= \frac{(k+1)^2}{4}(k+2)^2.
\end{aligned}$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

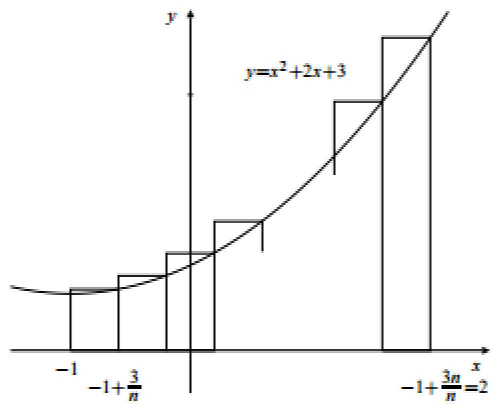

Figure 1: 3a

3. (a) The required area is (see the figure $3a$)

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{3}{n}[(-1+\frac{3}{n})^2 + 2(-1+\frac{3}{n}) + 3 \\ &\quad +(-1+\frac{6}{n})^2 + 2(-1+\frac{6}{n}) + 3 \\ &\quad +\cdots+(-1+\frac{3n}{n})^2 + 2(-1+\frac{3n}{n}) + 3] \\ &= \lim_{n\to\infty} \frac{3}{n}[(1-\frac{6}{n}+\frac{3^2}{n^2}-2+\frac{6}{n}+3) \\ &\quad +(1-\frac{12}{n}+\frac{6^2}{n^2}-2+\frac{12}{n}+3) \\ &\quad +\cdots+(1-\frac{6n}{n}+\frac{9n^2}{n^2}-2+\frac{6n}{n}+3)] \\ &= \lim_{n\to\infty}(6+\frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}) \\ &= 6+9 = 15 sq.units. \end{aligned}$$

(b) The height of the region at position x is $0-(x^2-2x) = 2x - x^2$. The base is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(2\frac{2i}{n} - \frac{4i^2}{n^2}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{8i}{n^2} - \frac{8i^2}{n^3}) \\ &= \lim_{n\to\infty} (\frac{8}{n^2}\frac{n(n+1)}{2} - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}) \\ &= 4 - \frac{8}{3} = \frac{4}{3} sq.units. \end{aligned}$$

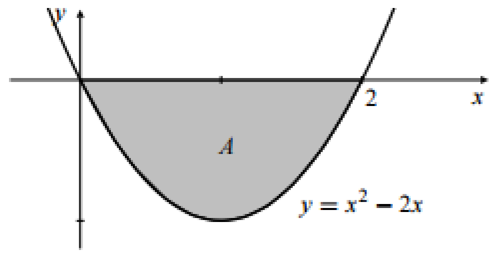

Figure 2: 3b

4.

$$P = \{x_0 < x_1 < \cdots < x_n\},$$
$$P' = \{x_0 < x_1 < \cdots < x_{j-1} < x' < x_j < \cdots < x_n\}.$$

Let m_j and M_j be, respectively, the minimum and maximum values of $f(x)$ on the interval $[x_{i-1}, x_i]$, for $1 \le i \le n$. Then

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

If m'_j and M'_j are the minimum and maximum values of $f(x)$ on $[x_{j-1}, x']$, and if m''_j and M''_j are the corresponding values for $[x', x_j]$, then

$$m'_j \ge m_j, \quad m''_j \ge m_j, \quad M'_j \le M_j, \quad M''_j \le M_j.$$

Therefore we have

$$m_j(x_j - x_{j-1}) \le m'_j(x' - x_{j-1}) + m''_j(x_j - x'),$$

$$M_j(x_j - x_{j-1}) \ge M'_j(x' - x_{j-1}) + M''_j(x_j - x'),$$

Hence $L(f,P) \le L(f,P')$ and $U(f,P) \ge U(f,P')$. If P'' is any refinement of P we can add the new points in P'' to those in P one at a time, and thus obtain

$$L(f,P) \le L(f,P''), \quad U(f,P) \ge U(f,P'').$$

5. (a)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2n+3i}{n^2} = \int_0^1 (2+3x)dx = \frac{7}{2}.$$

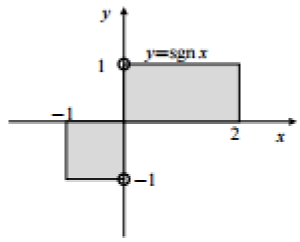

Figure 3: 5c

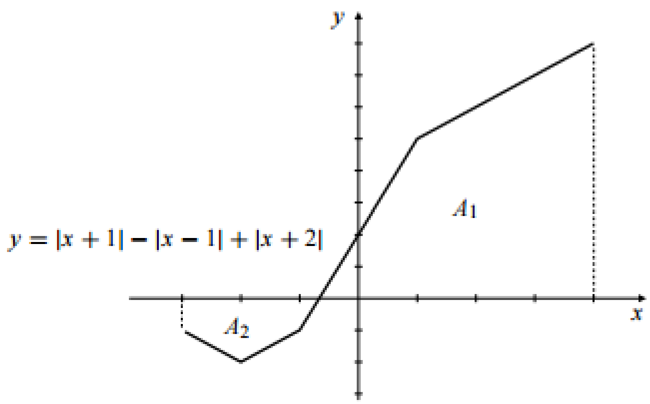

Figure 4: 5d

(b)

$$\begin{aligned}\int_{-3}^{3}(2+t)\sqrt{9-t^2}dt &= 2\int_{-3}^{3}\sqrt{9-t^2}dt + \int_{-3}^{3}t\sqrt{9-t^2}dt \\ &= 2(\frac{1}{2}\pi 3^2) + 0 = 9\pi.\end{aligned}$$

(c)

$$\int_{-1}^{2} sgn(x)dx = 2 - 1 = 1.$$

(d)

$$\begin{aligned}&\int_{-3}^{4}(|x+1| - |x-1| + |x+2|)dx \\ =\ & area(A_1) - area(A_2) \\ =\ & \frac{1}{2}\frac{5}{3}(5) + \frac{5+8}{2}(3) - \frac{1+2}{2}(1) - \frac{1+2}{2}(1) - \frac{1}{2}\frac{1}{3}(1) = \frac{41}{2}.\end{aligned}$$

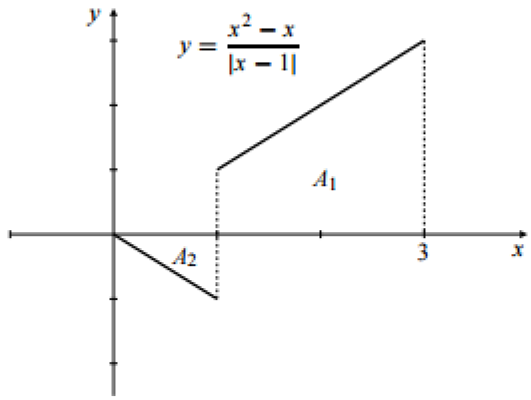

Figure 5: 5e

(e)

$$\begin{aligned} & \int_0^3 \frac{x^2 - x}{|x-1|} dx \\ = & \ area(A_1) - area(A_2) \\ = & \ \frac{1+3}{2}(2) - \frac{1}{2}(1)(1) = \frac{7}{2} \end{aligned}$$

(f)

$$\begin{aligned} & \int_a^b (f(x) - k)^2 dx \\ = & \int_a^b (f(x))^2 dx - 2k \int_a^b f(x) dx + k^2 \int_a^b dx \\ = & \int_a^b (f(x))^2 dx - 2k(b-a)\bar{f} + k^2(b-a) \\ = & \ (b-a)(k-\bar{f})^2 + \int_a^b (f(x))^2 dx - (b-a)(\bar{f})^2 \end{aligned}$$

This is minimum if $k = \bar{f}$.