Math 121: Homework 2 solutions

1.

$$\begin{aligned} 2f(x)+1 &= 3\int_x^1 f(t)dt \\ 2f'(x) &= -3f(x) \\ f(x) &= Ce^{-3x/2}, \end{aligned}$$

since $2f(x)+1=0$, $f(1)=-\frac{1}{2}=Ce^{-3/2}$, $C=-\frac{1}{2}e^{3/2}$.

$$f(x)=-\frac{1}{2}e^{(3/2)(1-x)}.$$

2. (a) The curves $y=\frac{4}{x^2}$ and $y=5-x^2$ intersect where $x^4-5x^2+4=0$. Thus the intersections are at $x=\pm1$ and $x=\pm2$. And the region is symmetric about $y-$axis. We have

$$\begin{aligned} Area &= 2\int_1^2 (5-x^2-\frac{4}{x^2})dx \\ &= 2\,(5x-\frac{x^3}{3}+\frac{4}{x})\Big|_1^2 = \frac{4}{3}sq.units. \end{aligned}$$

(b)

$$Looparea = 2\int_{-2}^0 x^2\sqrt{2+x},$$

Let $u^2=2+x$, $2udu=dx$. Thus we have

$$\begin{aligned} Area &= 2\int_0^{\sqrt{2}} (u^2-2)^2 u(2u)du \\ &= 4\int_0^{\sqrt{2}} (u^6-4u^4+4u^2)du \\ &= 4\,(\frac{1}{7}u^7-\frac{4}{5}u^5+\frac{4}{3}u^3)\Big|_0^{\sqrt{2}} \\ &= \frac{256\sqrt{2}}{105}sq.units. \end{aligned}$$

3. (a)

$$I=\int_0^4 \sqrt{9t^2+t^4}dt=\int_0^4 t\sqrt{9+t^2}dt,$$

let $u=9+t^2$, $du=2tdt$,

$$\begin{aligned} I &= \frac{1}{2}\int_9^{25}\sqrt{u}du \\ &= \frac{1}{3}\,u^{3/2}\Big|_9^{25} \\ &= \frac{98}{3}. \end{aligned}$$

(b)

$$\begin{aligned} I &= \int \cos^2(\frac{t}{5}) \sin^2(\frac{t}{5}) dt \\ &= \frac{1}{4} \int \sin^2(\frac{2t}{5}) dt \\ &= \frac{1}{8} \int (1 - \cos(\frac{4t}{5})) dt \\ &= \frac{1}{8}(t - \frac{5}{4} \sin(\frac{4t}{5})) + C. \end{aligned}$$

(c)

$$\begin{aligned} \int \cos^4 x dx &= \int \frac{[1 + cos(2x)]^2}{4} dx \\ &= \frac{1}{4} \int [1 + 2\cos(2x) + \cos^2(2x)] dx \\ &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) dx \\ &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C \\ &= \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C \end{aligned}$$

(d)

$$I = \int \frac{dx}{e^x + 1} = \int \frac{e^{-x} dx}{1 + e^{-x}},$$

let $u = 1 + e^{-x}$, $du = -e^{-x} dx$. Thus we have

$$\begin{aligned} I &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln(1 + e^{-x}) + C. \end{aligned}$$

(e)

$$Area = \int_0^2 \frac{x dx}{x^4 + 16}.$$

Let $u = x^2$, $du = 2x dx$. Thus, we have

$$\begin{aligned} Area &= \frac{1}{2} \int_0^4 \frac{du}{u^2 + 16} \\ &= \frac{1}{8} \tan^{-1} \frac{u}{4} \Big|_0^4 \\ &= \frac{\pi}{32} sq.units. \end{aligned}$$

(f)

$$I=\int_0^{\pi/2}\sqrt{1-\sin(\theta)}d\theta=\int_0^{\pi/2}\sqrt{1-\cos(\pi/2-x)}dx.$$

Let $u=\pi/2-x$, $du=-dx$.

$$\begin{aligned} I &= -\int_{\pi/2}^{0}\sqrt{1-\cos u}du \\ &= \int_0^{\pi/2}\sqrt{2\sin^2\frac{u}{2}}du=\sqrt{2}\,(-2\cos\frac{u}{2})\Big|_0^{\pi/2} \\ &= -2+2\sqrt{2}=2(\sqrt{2}-1). \end{aligned}$$

4. We want to prove that for each positive integer k,

$$\sum_{j=1}^{n}j^k=\frac{n^{k+1}}{k+1}+\frac{n^k}{2}+P_{k-1}(n),$$

where P_{k-1} is a polynomial of degree at most $k-1$. First check the case $k=1$:

$$\sum_{j=1}^{n}j=\frac{n(n+1)}{2}=\frac{n^{1+1}}{1+1}+\frac{n}{2}+P_0(n),$$

where $P_0(n)=0$ has degree ≤ 0. Now assume that the formula above holds for $k=1,2,3,\cdots,m$, we will show that it also holds for $k=m+1$. TO this end, sum the formula

$$(j+1)^{m+2}-j^{m+2}=(m+2)j^{m+1}+\frac{(m+2)(m+1)}{2}j^m+\cdots+1,$$

for $j=1,2,\cdots,n$. The left side telescopes, we get

$$\begin{aligned} (n+1)^{m+2}-1 &= (m+2)\sum_{j=1}^{n}j^{m+1} \\ &+ \frac{(m+2)(m+1)}{2}\sum_{j=1}^{n}j^m+\cdots+\sum_{j=1}^{n}1. \end{aligned}$$

Expanding the binomial power on the left and using the induction hypothesis on the other terms we get

$$\begin{aligned} n^{m+2}+(m+2)n^{m+1}+\cdots &= (m+2)\sum_{j=1}^{n}j^{m+1} \\ &+ \frac{(m+2)(m+1)}{2}\frac{n^{m+1}}{m+1}+\cdots, \end{aligned}$$

where the $\cdots$ represent terms of degree m or lower in the variable n. Solving for the remain sum, we get

$$\begin{aligned}\sum_{j=1}^{n} j^{m+1} &= \frac{1}{m+2}(n^{m+2} + (m+2)n^{m+1} + \cdots - \frac{m+2}{2}n^{m+1} - \cdots) \\ &= \frac{n^{m+2}}{m+2} + \frac{n^{m+1}}{2} + \cdots,\end{aligned}$$

so that the formula is also correct for $k = m+1$. Hence it is true for all positive integer k by induction.

5.

$$F(x) = \int_0^{2x-x^2} \cos(\frac{1}{1+t^2})dt.$$

Note that $0 < \frac{1}{1+t^2} \leq 1$ for all t, and hence

$$0 < \cos(1) \leq \cos(\frac{1}{1+t^2}) \leq 1.$$

The integrand is continuous for all t, so $F(x)$ is defined and differentiable for all x. Since $\lim_{x\to\pm\infty}(2x - x^2) = -\infty$, therefore $\lim_{x\to\pm\infty} F(x) = -\infty$. Now

$$F'(x) = (2-2x)\cos(\frac{1}{1+(2x-x^2)^2}) = 0$$

only at $x = 1$. Therefore F must have a maximum value at $x = 1$, and no minimum value.

6. (a) If m and n are integers, and $m \neq n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m-n)x + \cos(m+n)x)dx \\ &= \frac{1}{2}(\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n})\Big|_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\cos(m-n)x - \cos(m+n)x)dx \\ &= \frac{1}{2}(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n})\Big|_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\sin(m+n)x + \sin(m-n)x)dx \\ &= -\frac{1}{2}(\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n})\Big|_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

If $m = n \neq 0$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(mx)dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\sin(2mx))dx \\ &= -\frac{1}{4m}\left(\cos(2mx)\right)\Big|_{-\pi}^{\pi} \\ &= 0,\end{aligned}$$

(b) If $1 \leq m \leq k$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)dx \\ &+ \sum_{n=1}^{k} a_n \int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx \\ &+ \sum_{n=1}^{k} b_n \int_{-\pi}^{\pi}\sin(nx)\cos(mx)dx.\end{aligned}$$

By the previous part, all the integrals on the right side are zero except the one in the first sum having $n = m$. Thus the whole right side reduces to

$$\begin{aligned}a_m \int_{-\pi}^{\pi}\cos^2(mx)dx &= a_m \int_{-\pi}^{\pi}\frac{1+\cos(2mx)}{2}dx \\ &= \frac{a_m}{2}(2\pi + 0) = \pi a_m.\end{aligned}$$

Thus

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx.$$

Similarly, we have

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx.$$

For $m = 0$ we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \int_{-\pi}^{\pi} f(x)dx \\ &= a_0\pi,\end{aligned}$$

so the formula for a_m also holds for $m = 0$.