THE UNIVERSITY OF BRITISH COLUMBIA
Sessional Examinations.April 2005

MATHEMATICS 121

Closed book examination **Time:2 1/2 hours**

Calculators are not allowed in this examination

I-[21] SHORT ANSWERS QUESTIONS Each question is worth 3 marks, but not all questions are of equal difficlty. Full marks will be given for correct answers, but at most one mark will be given for incorrect answers. Simplify your answers as much as possible.

a) Evaluate $\int (x^2 + e^{2x})dx$.

b) Find the average value of $\sin x$ on rthe interval $[0, \pi]$.

c) Find $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} \sin^2(j/n)$.

d) Find the general solution $y = y(x)$ of the differential equation $y'' - 2y' + y = 0$.

e) Find the general solution $y = y(x)$ of the differential equation $y'' - 2y' + y = x$.

f) Evaluate $\int_0^{\infty} (1+x)^{-3} dx$.

g) A continuous random variable X is exponentially distributed with a mean of 4. Find the probability that $X \geq 8$.

In Questions II-IX, justify your answers and **show all your work**. Unless otherwise indicated, simplification of answers is not required.

II-15] Let R be the finite region bounded above by the curve $y = 4 - x^2$ and below by $y = 2 - x$.

a)-[**4**] Carefully sketch R and find its area explicitly.

b)-[3] Express the volume of the solid obtained by rotating R about the x-axis as a definite integral. You **do not** need to evaluate this integral.

c)-[4] Express the volume of the solid obtained by rotating R about the vertical line $x = 2$ as a definite integral. You **do not** need to evaluate this integral

d)-[4] Express the length of the upper curve that bounds R as a definite integral, and using an appropriate substitution express your answer as an integral involving trigonometric functions. You **do not** need to evaluate this trigonometric integral.

III-[16] Evaluate the following integrals.

a)-[5] $\int \frac{4x+4}{x(x+1)^2}dx$.

b)-[4] $\int (x+1)\ln x dx$.

c)-[7] $\int \frac{dx}{(5-4x-x^2)^{3/2}}$.

IV-[8] Find $f(x)$ such that

$$f(x) = 1 + \int_0^x \frac{tf(t)}{1+t+t^2}dt.$$

V-[8] A child initially at O walks along the edge of a pier, towing a sailboat by a string of length L. The pier is taken to be the y axis.

(a) If the boat starts at Q and the string always remains straight, show that the equation of the curved path $y = f(x)$ followed by the boat must satisfy the differential equation

$$\frac{dy}{dt} = -\frac{\sqrt{L^2 - x^2}}{x}$$

(b) Find $y = f(x)$.

VI-[8] The vertical face of a dam across a river has the shape of a parabola 36m across the top and 9m deep at the center. What is the force that the river exerts on the dam if the water is 0.5m from the top? Density of water is $1,000 kg/m^3$.

VII-[**8**] Solve the initial value problem

$$y'' + y = \sec^3 x, \quad y(0) = 1, \; y'(0) = 0.$$

VIII-[**8**]The length of time in minutes it takes students to solve a certain mathematics problem (on probability) is a continuous random variable whose probability density function is

$$f(x) = \begin{cases} k \cos(x/10) \sin^2(x/10) & \text{if } 0 \leq x \leq 5\pi \\ 0 & \text{otherwise} \end{cases}$$

a)-[**2**] Find the value of the positive constant k.

b)-[**3**]What is the probability that a student will take less than $5\pi/2$ minutes to complete the problem?

c)-[**3**] Compute the mean length of time requires to solve the problem.

IX-8] Let $a_1 = 2$ and

$$a_{n+1} = \frac{2 + a_n}{1 + a_n}, \quad n = 1, ...$$

a)-[**5**] Show that

$$1)....a_{2n+1} < a_{2n-1} < ... < a_3 < a_1$$

and that

$$2) \;\; a_2 < a_4 < ... < a_{2n-2} < a_{2n} < ....$$

b)-[**3**] Prove that the sequences $\{a_{2n}\}$ and $\{a_{2n+1}$ converge.

Bonus-[**4**] Show that the sequence a_n converges and that

$$\lim_{n \to \infty} a_n = \sqrt{2}.$$