Math 121 Assignment 4
Due Friday February 5

1. For each of the two integrals below, either evaluate the integral or show that it diverges.

$$(a)\ \int_0^1 \frac{dx}{\sqrt{x(1-x)}} \qquad (b)\ \int_0^{\frac{\pi}{2}} \sec x\, dx$$

2. State whether the given integral converges or diverges, and justify your claim.

$$(a)\ \int_0^{\pi^2} \frac{dx}{1-\cos(\sqrt{x})} \quad (b)\ \int_0^{\infty} \frac{|\sin x|}{x^2}\, dx \quad (c)\ \int_2^{\infty} \frac{dx}{\sqrt{x}\ln x}.$$

3. Rewrite the integrals below in a form to which numerical methods can be readily applied. Do not evaluate or find numerical approximations for the integrals.

$$(a)\ \int_1^{\infty} \frac{dx}{x^2+\sqrt{x}+1} \quad (b)\ \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} \quad (c)\ \int_{-1}^{1} \frac{e^x\, dx}{\sqrt{1-x^2}}.$$

4. Transform the integral $I = \int_1^\infty e^{-x^2}\, dx$ using the substitution $x = 1/t$ and calculate the Simpson's rule approximations S_2, S_4 and S_8 for the resulting integral.
5. The goal of this problem is to prove an error estimate for the midpoint rule. We will use the error formula for the tangent line approximation. Namely, given a function f with a continuous second derivative on $[a,b]$ and satisfying the bound $|f''(x)| \le K$ there, assume the inequality

$$|f(x) - f(m_1) - f'(m_1)(x-m_1)| \le \frac{K}{2}(x-m_1)^2 \text{ for } x \in [x_0, x_1].$$

where x_0, x_1 are any two points in $[a,b]$ and m_1 is the midpoint of $[x_0, x_1]$.
(a) Use this inequality to show that

$$\left| \int_{x_0}^{x_1} f(x)\, dx - f(m_1)h \right| \le \frac{K}{24}h^3,$$

and use the estimate above to prove the following error bound for the midpoint rule:

$$\left| \int_a^b f(x)\, dx - M_n \right| \le \frac{K(b-a)^3}{24n^2}.$$

(b) Compute M_1 for the function $f(x) = x^2$ with $a = 0$ and $b = 1$ to show that the error bound obtained in part (a) cannot be improved in general.

6. The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) Show that the integral converges for $x > 0$.

(b) Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

(c) Show that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

(d) If you are given that $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$, show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$.