Homework 1 - Math 541, Spring 2016

Due February 12 at the beginning of the lecture

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. Let M denote the Hardy-Littlewood maximal function. Show that if f is not identically zero, then Mf is never integrable on $\mathbb{R}^n$.

2. We proved in class that for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the family of averages

$$\frac{1}{|B(x;r)|} \int_{B(x;r)} f(y)\, dy \tag{1}$$

admits a limit for almost every $x \in \mathbb{R}^n$ as $r \to 0$. Modify that argument to show that, in fact, the complement of the set

$$L(f) = \Big\{ x \in \mathbb{R}^n : \lim_{r\to 0} \frac{1}{|B(x;r)|} \int_{B(x;r)} |f(y) - f(x)|\, dy = 0 \Big\} \tag{2}$$

is of null Lebesgue measure. Hence deduce that for almost every x, the limit of (1) as $r \to 0$ is $f(x)$. The set in (2) is called the *Lebesgue set* of f.

3. For each of the criteria specified below, find an example of a Borel set $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ with this property.
   (a) the limiting value of the averages in (1) does not exist with $f = \mathbf{1}_E$ as $r \to 0$.
   (b) Given any number $\alpha \in (0,1)$ and $f = \mathbf{1}_E$, the limiting value exists and equals α.

4. Let μ be any regular complex measure on $\mathbb{R}^n$. Discuss the limiting behaviour, as $r \to 0$, of the averages $\mu(B(x;r)/|B(x;r)|$, possibly excluding a class of points x of zero Lebesgue measure.

5. Let $\mathcal{S}$ be a family of measurable sets in $\mathbb{R}^n$ with the following property: for each $x \in \mathbb{R}^n$ and $r > 0$, there exists $S_r(x) \in \mathcal{S}$ satisfying

$$S_r(x) \subseteq B(x;r) \qquad \text{and} \qquad |B(x;r)| \leq C|S_r(x)|,$$

for some constant $C > 0$ independent of r and x.

(a) Give at least two distinct examples of families of sets $\mathcal{S}$ that meets the two requirements described above. Also provide at least two examples of $\mathcal{S}$ which satisfies the first condition but does not meet the second.

(b) Show that

$$\lim_{r\to 0} \frac{1}{|S_r(x)|} \int_{S_r(x)} |f(y) - f(x)|\, dy = 0$$

for every point x in the Lebesgue set of f.

6. The Hardy Littlewood maximal operator M is of fundamental importance in part because it controls many other operators of interest arising in a variety of contexts. We illustrate this in the context of the Dirichlet problem for Laplace's equation.

(a) Suppose that $g : \mathbb{R}^d \to [0, \infty]$ is radial and nonincreasing. In other words, $g(x) = h(|x|)$ with $h(r_1) \geq h(r_2)$ for $0 \leq r_1 \leq r_2$. Show that $f * g(x) \leq ||g||_1 Mf(x)$ for all x and all non-negative f.

(b) Recall the Poisson kernel for the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$:

$$p_t(x) = c_n t^{-n}(1 + |t^{-1}x|^2)^{-\frac{n+1}{2}}.$$

Verify that for any bounded continuous f or for $f \in L^p$, $p \in [1, \infty]$, the function $u(x,t) = f * p_t(x)$ obeys Laplace's equation $\triangle u = 0$ on $\mathbb{R}^{n+1}_+$.

(c) Let's focus now on the boundary behaviour of u. Show that $u(x,t) \to f(x)$ as $t \to 0$ uniformly on compact sets if f is a bounded continuous function. Prove convergence in L^p as $t \to 0$ if $f \in L^p(\mathbb{R}^n)$.

(d) What can we say about the pointwise convergence of u to f? Show that $u(x,t) \to f(x)$ as $t \to 0$ *non-tangentially* for almost every $x \in \mathbb{R}^n$. This means that for almost every x, and every $r > 0$,

$$u(y,t) \to f(x) \quad \text{as } (y,t) \to (x,0), \text{ with}$$
$$(y,t) \in \Gamma_r(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| < rt\}.$$