

Homework 2 - Math 541, Spring 2016

Due on Friday February 26

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

1. Complete the following outline to show that for $1 < p < \infty$, then there exists a genuine norm $|\cdot|_{p,\infty}$ on $L^{p,\infty}$ equivalent to the quasinorm $\|\cdot\|_{p,\infty}$. For each function f that is absolutely integrable on every set of finite measure, define

$$|f|_{p,\infty} = \sup_E \mu(E)^{-\frac{1}{p}} \int_E |f| d\mu,$$

the supremum being taken over all sets satisfying $\mu(E) < \infty$.

- (a) Verify that $|\cdot|_{p,\infty}$ is a norm, and that $L^{p,\infty}(X)$ is complete under the norm.
- (b) Show that if $1 < p < \infty$, then $|f|_{p,\infty} \sim \|f\|_{p,\infty}$, that is, both are finite if either is, and each is bounded by a (p -dependent) constant multiple of the other, uniformly in f .
2. Prove the following extension of Young's convolution inequality, and its consequence, fractional integration.
- (a) If $p, q, r \in (1, \infty)$ and $r^{-1} = p^{-1} + q^{-1} - 1$, then $L^p(\mathbb{R}^d) * L^{q,\infty}(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$.
- (b) The operator $f \mapsto f * |x|^{-\frac{d}{q}}$ maps $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ whenever $1 < p, q, r < \infty$ and $r^{-1} = p^{-1} + q^{-1} - 1$, $p^{-1} + q^{-1} - 1$.
3. Give an example to show that the conclusion of the three-lines lemma fails without any assumption on the growth of the analytic function at ∞ .
4. Verify whether the following statement is true or false. "Suppose that $T : \mathcal{C}_b(\mathbb{R}^d) \rightarrow \mathcal{C}_b(\mathbb{R}^d)$ is a bounded linear operator that commutes with translations. Then there exists a measure μ on \mathbb{R}^d such that $Tf = f * \mu$ for all $f \in \mathcal{C}_b$." Here $\mathcal{C}_b(\mathbb{R}^d)$ denotes the space of all complex-valued bounded continuous functions on \mathbb{R}^d .
5. Show that the algebra $L^1(\mathbb{R}^d)$ has no multiplicative unit.
6. Define the convolution measure of two finite measures μ and ν , and show that the total variation of $\mu * \nu$ satisfies $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

7. We had an informal discussion in class about the metaprinciple that convolutions tend to smooth functions out. The following results, which you should prove, are a few instances of this general philosophy.

(a) Fix any $p \in [1, \infty]$. Given any $f \in L^p$ and $g \in L^{p'}$, their convolution $f * g$ is a continuous (!) function, vanishing at infinity (!) if $p \in (1, \infty)$. Here p and p' denote dual exponents, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

(b) Let $1 \leq p \leq \infty$, $\gamma > d$ and $k \geq 0$. Suppose that $\varphi \in \mathcal{C}_b^k(\mathbb{R}^d)$ with

$$|\partial^\alpha \varphi(x)| \leq C(1 + |x|)^{-\gamma} \quad \text{for all } x \text{ and all } |\alpha| \leq k.$$

Then for any $f \in L^p$, one has that $f * \varphi \in \mathcal{C}_b^k$, with $\partial^\alpha(f * \varphi) = f * \partial^\alpha \varphi$ for all α with $|\alpha| \leq k$. Moreover, $\|f * \varphi\|_{\mathcal{C}_b^k} \leq C\|f\|_p$ for some constant C independent of f . Here $\mathcal{C}_b^k(\mathbb{R}^d)$ denotes the space of k -times continuously differentiable functions on \mathbb{R}^d , equipped with the norm $\|g\|_{\mathcal{C}_b^k(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} |\partial^\alpha g(x)|$.

(c) If the function f in part (b) is chosen to be in \mathcal{C}_b^ℓ instead, show that $f * \varphi \in \mathcal{C}_b^{k+\ell}$.

8. Let $\{\varphi_j\} \subseteq L^1(\mathbb{R}^d)$ be an approximate identity family, i.e., obeys the following three properties:

$$(i) \int \varphi_j \xrightarrow{j \rightarrow \infty} 1, \quad (ii) \sup_j \|\varphi_j\|_1 < \infty, \quad (iii) \int_{|x| > \delta} |\varphi_j(x)| dx \xrightarrow{j \rightarrow \infty} 0 \quad \forall \delta > 0.$$

We showed in class that $f * \varphi_j \rightarrow f$ uniformly if $f \in \mathcal{C}_c(\mathbb{R}^d)$ and in L^p -norm if $f \in L^p$.¹ To what extent do these convergence results characterize the three properties (i)-(iii) of an approximate identity family? We try to explore this converse direction in the following sequence of problems.

(a) Suppose that $\{\psi_j : j \geq 1\}$ is a subset of $L^1(\mathbb{R}^d)$ satisfying $\|f * \psi_j - f\|_1 \rightarrow 0$ as $j \rightarrow \infty$ for every $f \in L^1$. Prove that $\int \psi_j \rightarrow 1$ as $j \rightarrow \infty$ and that $\|\psi_j\|_1$ is bounded uniformly in j .

(b) Show that there exist sequences $\{\psi_j\} \subseteq L^1(\mathbb{R}^d)$ such that $f * \psi_j \rightarrow f$ uniformly for all $f \in \mathcal{C}_c(\mathbb{R}^d)$, yet $\{\psi_j\}$ does not satisfy part (iii) of the definition of an approximate identity sequence.

(c) Show that there also exist such sequences satisfying $\|f * \psi_j - f\|_1 \rightarrow 0$ for all $f \in L^1$.

¹We assumed that $\int \varphi_j = 1$ for all j , but convince yourself that the proofs go through with minimal modifications if this requirement is replaced by the milder (i) above.