Homework 3 - Math 541, Spring 2016

Due Friday March 18.

Instructions: Your homework will be graded both on mathematical correctness and quality of exposition. Please pay attention to the presentation of your solutions.

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1. *Evaluation of the Fourier transform for some examples.* Establish the following identities:
   (a) For any $\xi \in \mathbb{R}$,
   $$\big[(1+x^2)^{-1}\big]\widehat{\ }(\xi) = \pi e^{-|\xi|}.$$
   (b) For any $\xi \in \mathbb{R}$,
   $$\Big[e^{-x^2/2}\Big]\widehat{\ }(\xi) = \sqrt{2\pi}e^{-\xi^2/2}.$$
   (c) In $\mathbb{R}^d$,
   $$\big[e^{-|\xi|}\big]\widehat{\ }(x) = (2\pi)^d\Gamma\Big(\frac{d+1}{2}\Big)\pi^{-\frac{d+1}{2}}\left(1+|x|^2\right)^{-\frac{d+1}{2}}.$$
   As part of the proof of Plancherel's theorem presented in class, we showed that there exists an absolute constant $\beta > 0$ such that $(\widehat{f})^{\vee} = \beta^{-d} f$ almost everywhere for every $d \geq 1$ and $f \in L^1(\mathbb{R}^d)$. Use any subset of the formulae above to find the value of β.
   (*Hint for (c):* Start by writing $e^{-|\xi|}$ as
   $$e^{-|\xi|} = \pi^{-\frac{1}{2}}\int_0^\infty e^{-\frac{|\xi|^2}{4u}}e^{-u}u^{-\frac{1}{2}}\,du,$$
   then evaluate the Fourier transform using (b).)

2. Prove that for $q < 2$, the Fourier transform does not map the set of all L^∞ functions supported on the unit ball of $\mathbb{R}^d$ to L^q. Use this to deduce that the Hausdorff-Young inequality is valid for no $p > 2$.

3. Recall the definition of temperate measure introduced in class.
   (a) Given any temperate measure μ and multi-index α, show that φ defined by
   $$\langle \varphi, f\rangle := \int_{\mathbb{R}^d}\partial^\alpha f(x)\,d\mu(x), \quad f \in \mathcal{S}(\mathbb{R}^d) \qquad (*)$$
   is a tempered distribution.
   (b) Show that every element of $\mathcal{S}'(\mathbb{R}^d)$ is a finite linear combination of distributions of the form $(*)$, for certain temperate measures μ and multi-indices α.

4. This problem deals with some more examples of tempered distributions.

(a) Show that the principal value $1/x$ distribution, defined by

$$\text{pv} \int f(x)x^{-1}\, dx := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} f(x)x^{-1}\, dx$$

is tempered. We sketched a proof of this in class, but fill in all the details.

(b) Show that $\varphi(\cdot)$, defined by

$$\langle \varphi(z), f \rangle = \int_0^\infty x^z f(x)\, dx$$

is a holomorphic $\mathcal{S}'$-valued function of z for $\mathrm{Re}(z) > -1$, which can be continued as a meromorphic $\mathcal{S}'$-valued function on all of $\mathbb{C}$, with only simple poles at $z = -1, -2, -3, \cdots$.

5. We studied a number of examples in class where certain classical notions, such as derivatives, Fourier transforms, multiplication by polynomials, were extended from the function-theoretic setting to that of tempered distributions. Here are some more. Give appropriate definitions of the following, making sure that your definitions are consistent with existing ones.

(a) $\psi \in \mathcal{S}'(\mathbb{R}^d)$ has compact support.

(b) $\varphi * f$, where $\varphi \in \mathcal{S}'$, $f \in \mathcal{S}$.

(c) $\varphi * \psi$, where $\varphi, \psi \in \mathcal{S}'$, ψ has compact support. Is $\varphi * \psi = \psi * \varphi$?

6. A sequence $\{a_\ell : \ell \geq 1\} \subseteq \mathbb{T}$ is said to be equidistributed if for every interval $I \subseteq \mathbb{T}$, $\#(\{1 \leq \ell \leq n : a_\ell \in I\})/n \to |I|/(2\pi)$ as $n \to \infty$. Weyl's equidistribution theorem states that a sequence $\{a_\ell : \ell \geq 1\}$ is equidistributed if and only if

$$n^{-1} \sum_{\ell=1}^{n} e^{ika_\ell} \to 0 \text{ as } n \to \infty, \qquad \text{for every integer } k \neq 0. \tag{1}$$

Fill in the following steps to prove Weyl's theorem.

(a) Assume that $\{a_\ell\}$ is equidistributed. Show that for any continuous function φ defined on $\mathbb{T}$,

$$n^{-1} \sum_{\ell=1}^{n} \varphi(a_\ell) \to (2\pi)^{-1} \int_{\mathbb{T}} \varphi(x)\, dx. \tag{2}$$

Specialize to characters e^{ikx} to prove half of Weyl's theorem.

(b) Assuming (1), use Fourier series to prove that (2) holds. Now approximate the characteristic function of an interval above and below by continuous functions to conclude the other half of the proof from (2).

7. Recall from our discussion in class that the function
$$k_t(x) = (2\pi)^{\frac{d}{2}}(2it)^{-\frac{d}{2}}e^{i|x|^2/(4t)}$$
satisfies Schrödinger's equation on $\mathbb{R}^{d+1}_+$.
   (a) Show that $u(x,t) = f * k_t$ is a continuous function if $f \in L^1(\mathbb{R}^d)$ and a measurable function if $f \in L^2(\mathbb{R}^d)$.
   (b) Show that in either case $(\partial_t - i\triangle)u = 0$ in the sense of distributions.
   (c) Describe in which sense $u(\cdot,t) \to f$ as $t \to 0+$, when $f \in L^1$ and also when $f \in L^2$.

8. Determine whether each of the following statements is true or false. Give brief justification for your answers.
   (a) For any $p \in [1,\infty)$, the set of all functions in $L^p(\mathbb{R}^d)$ whose Fourier transforms have compact support is dense in $L^p(\mathbb{R}^d)$.
   (b) There exists $p \neq 2$ such that the Fourier transform can be extended as a bounded linear map from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$.
   (c) For any $f \in L^1(\mathbb{R}^d)$, $\widehat{f}(\xi)$ is continuous and $\to 0$ as $|\xi| \to \infty$.
   (d) Every continuous function on $\mathbb{R}^d$ vanishing at infinity is the Fourier transform of an L^1 function.
   (e) There exists $f \in L^\infty \setminus \cup_{1\le p<\infty} L^p$ whose Fourier transform is a function, not merely a distribution.
   (f) If $f \in C(\mathbb{T}^d)$ satisfies a Hölder condition with exponent α, $0 < \alpha \le 1$, then $\widehat{f}(n) = O(|n|^{-\alpha})$.