MATH 421/510
Assignment 2

Suggested Solutions

February 2018

1. Let X be an infinite-dimensional Banach space. Show that every Hamel basis of X is uncountable.

Proof. Our idea is to use the Baire Category theorem.

Suppose there were a countable Hamel basis for X, given by $B := \{x_1, x_2, \dots\}$. Let $X_n := \text{span}\{x_1, x_2, \dots, x_n\}$, which is closed in X, since it is a finite dimensional subspace. Furthermore, since X_n is a proper subspace of X, it has no interior points. Therefore X_n is nowhere dense. Since $X = \cup_{n=1}^{\infty} X_n$, the Baire Category theorem gives rise to a contradiction.

□

2. (a) Show that the vector space of polynomials is dense in $C[0, 1]$, but the monomials $\{x^n : n \geq 1\}$ do not form a Schauder basis for $C[0, 1]$.
   (b) Does $C[0, 1]$ have a Schauder basis?

Proof. (a) That the vector space of polynomials is dense in $C[0, 1]$ is exactly the statement of the Weierstrass approximation theorem.

The monomials $\{x^n : n \geq 1\}$ do not form a Schauder basis for $C[0, 1]$. Indeed, if they did, then given any $f \in C[0, 1]$, there is a unique representation $f = \sum_{n=0}^{\infty} a_n x^n$ as a uniformly convergent power series whose radius of convergence is at least 1. This implies that f is differentiable at any $x \in [0, 1)$, which is not always the case if we pick, say, $f(x) = |x - \frac{1}{2}|$.

(b) i. **Construction of the system:**

This is a standard example called the Faber-Schauder system: $f_0(x) = 1$, and

$$f_{j,k} = (1 - 2^j|x - k/2^j|)^+, \quad j \geq 0, \quad 1 \leq k \leq 2^j, \quad k \text{ is odd}$$

We can arrange them in the natural way as $\{f_n\} := (f_0, f_{0,1}, f_{1,1}, f_{2,1}, f_{2,3}, \dots)$. You can refer to the following website for some pictures:
https://math.stackexchange.com/questions/667251/example-of-a-basis-of-c0-1

We present an idea of how we compute the unique expansion

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{1\le k\le 2^j, k \text{ odd}} c_{j,k} f_{j,k}(x).$$

We look into the values $f(\frac{i}{2^l})$ at the dyadic integers. This gives rise to the following system of linear equations:

$$\begin{cases} f(0) = c_0 \\ f(1) = c_0 + c_{0,1} \\ f\left(\frac{1}{2}\right) = c_0 + \frac{1}{2}c_{0,1} + c_{1,1}f_{1,1}\left(\frac{1}{2}\right) = c_0 + \frac{1}{2}c_{0,1} + c_{1,1} \\ f\left(\frac{1}{4}\right) = c_0 + \frac{1}{4}c_{0,1} + c_{1,1}f_{1,1}\left(\frac{1}{4}\right) + c_{2,1}f_{2,1}\left(\frac{1}{4}\right) + c_{2,3}f_{2,3}\left(\frac{1}{4}\right) = c_0 + \frac{1}{4}c_{0,1} + \frac{1}{2}c_{1,1} + c_{2,1} \\ f\left(\frac{3}{4}\right) = c_0 + \frac{3}{4}c_{0,1} + c_{1,1}f_{1,1}\left(\frac{3}{4}\right) + c_{2,1}f_{2,1}\left(\frac{3}{4}\right) + c_{2,3}f_{2,3}\left(\frac{3}{4}\right) = c_0 + \frac{3}{4}c_{0,1} + \frac{1}{2}c_{1,1} + c_{2,3} \\ \dots \end{cases}$$

We can thus solve for all the c_0, $c_{j,k}$ using forward substitutions.

ii. **Proof of Convergence:**
We constructed for each N, a piecewise (more specifically, in each $[\frac{k}{2^N}, \frac{k+1}{2^N}]$) linear function

$$p_N := c_0 + \sum_{j=0}^{N} \sum_{1\le k\le 2^j, k \text{ odd}} c_{j,k} f_{j,k}(x).$$

Moreover, p_N agrees with f at all dyadic integers by construction. We show that $p_N \to f$ uniformly.
Let $\varepsilon > 0$. Since $f \in C[0,1]$, it is uniformly continuous. Take $N_0 \in \mathbb{N}$ such that for $|x-y| \le 2^{-N_0}$, $|f(x) - f(y)| < \varepsilon$.
Then given $x \in [0,1]$ and $N \ge N_0$, there are dyadic numbers $y := \frac{k}{2^N}$, $z := \frac{k+1}{2^N}$ with $y \le x \le z$. This choice is made so that $|p_N(x) - p_N(y)| \le |p_N(y) - p_N(z)|$ since p_N is piecewise linear.
Thus we have: for all $N \ge N_0$ and $x \in [0,1]$,

$$\begin{aligned} |p_N(x) - f(x)| &\le |p_N(x) - p_N(y)| + |p_N(y) - f(y)| + |f(y) - f(x)| \\ &\le |p_N(y) - p_N(z)| + |p_N(y) - f(y)| + |f(y) - f(x)| \\ &= |f(y) - f(z)| + |f(y) - f(y)| + |f(y) - f(x)| \\ &< 2\varepsilon. \end{aligned}$$

iii. **Uniqueness of the Representation:**
One can see uniqueness trivially holds since the constants $c_{j,k}$ are chosen in a deterministic way. The following is a more rigorous argument:
Suppose there are two different expansions for f: $f = \sum_{n=0}^{\infty} c_n f_n = \sum_{n=0}^{\infty} d_n f_n$, given by the Faber-Schauder system. Let $N \ge 0$ be the least integer such that $c_n \ne d_n$. Then subtraction gives $\sum_{n=N}^{\infty}(c_n - d_n)f_n = 0$. By evaluating at the dyadic integer $\frac{k}{2^M}$ where $f_N = f_{M,k}$ for some k, we see that

$$0 = \sum_{n=N}^{\infty}(c_n - d_n)f_n\left(\frac{k}{2^M}\right) = c_N - d_N,$$

which is a contradiction. Hence the representation is unique.

□

3. (a) Let X be a normed space, and Y be a proper subspace of X (actually, the following holds trivially if $X = Y$). Denote X^* the space of all bounded linear functionals on X. Show that if $l \in Y^*$, then there exists $L \in X^*$ such that $L|_Y \equiv l$ and $\|L\| = \|l\|$.

   (b) Use the above to show that if X is a normed space and $x \in X$, then

$$\|x\| = \sup\{|l(x)| : l \in X^* \quad \text{and} \quad \|l\| \leq 1\}.$$

Solution. (a) We define a natural sublinear functional ϕ by $\phi(x) = \|l\|\|x\|$. For $y \in Y$, we have $|l(y)| \leq \|l\|\|y\| = \phi(y)$ by definition of operator norm. By the Hahn-Banach theorem, ϕ can be extended to X, which we denote as L, and $L(x) \leq \phi(x)$ for all $x \in X$.

It suffices to show that $\|L\| = \|l\|$. Indeed, as an extension, it is obvious that $\|L\| \geq \|l\|$. On the other hand, since $L(x) \leq \phi(x) = \|l\|\|x\|$, and $-L(x) = L(-x) \leq \phi(-x) = \|l\|\|x\|$ for all $x \in X$, the definition of operator norm shows that $\|L\| \leq \|l\|$. Therefore $\|L\| = \|l\|$.

(b) By definition of operator norm, it is trivial that

$$\|x\| \geq \sup\{|l(x)| : l \in X^* \quad \text{and} \quad \|l\| \leq 1\}.$$

To show the reverse inequality, given x, we let $Y := \text{span}\{x\}$, which is a subspace of X. Define a linear operator on Y by $k(y) := c\|x\|$ where $y = cx$. Since $x \neq 0$ and $\dim(Y) \leq 1$, k is well defined and linear. It is bounded since $|k(y)| = |c|\|x\| = \|y\|$ for all $y \in Y$, and thus $\|k\| \leq 1$.

By the first part of the question, we can extend k to $l \in X^*$ with $\|l\| = \|k\| \leq 1$. Moreover, since $x \in Y$, we have $|l(x)| = |k(x)| = \|x\|$. This shows that

$$\|x\| \leq \sup\{|l(x)| : l \in X^* \quad \text{and} \quad \|l\| \leq 1\}.$$

4. Let Y be a proper closed subspace of X, $u \in X \backslash Y$ and $\rho = \text{dist}(u, Y)$. Show that there exists a linear functional $l \in X^*$ such that $l(u) = 1$, $l \equiv 0$ on Y, and $\|l\| = \rho^{-1}$.

Proof. Let $u \in X \backslash Y$. Then $u \neq 0$ and $\rho = \text{dist}(u, Y) > 0$. Define a linear functional $k : \text{span}\{u\} \to \mathbb{F}$ by $k(cu) = c$. This map is well defined and linear, since $\dim(\text{span}\{\text{u}\}) = 1$.

Consider the function $p(x) := \rho^{-1}\text{dist}(x, Y)$. It is sublinear, and for all $cu \in Y$,

$$p(cu) = \rho^{-1}\text{dist}(cu, Y) \geq \rho^{-1}|c|\,\text{dist}(u, Y) \geq \rho^{-1}|c|\rho = |c| = |k(cu)|.$$

Hence we can apply the Hahn-Banach theorem to extend k to $l \in X^*$, with $|l(x)| \leq p(x)$. We can check that $|l(y)| \leq p(y) = 0$ for all $y \in Y$. Since $l(u) = 1$, it remains to show $\|l\| = \rho^{-1}$.

On one hand, since $0 \in Y$, $\operatorname{dist}(x, Y) \leq \|x\|$, and so

$$|l(x)| \leq p(x) \leq \rho^{-1}\|x\|.$$

Thus $\|l\| \leq \rho^{-1}$. On the other hand, by definition of the distance, there exists a sequence $y_n \in Y$ such that $\|u - y_n\| < +\rho + 1/n$. Noticing that

$$\|l\|\|u - y_n\| \geq l(u - y_n) = l(u) - l(y_n) = 1 - 0 = 1,$$

we have $\|l\| \geq \|u - y_n\|^{-1}$. Letting $n \to \infty$, we have $\|l\| \geq \rho^{-1}$. □

5. Show that there exists a linear functional l of norm 1 on the space of real bounded sequences that generalises the concept of limits, in the following sense:

   - l is shift invariant, that is, $l(x_1, x_2, \dots,) = l(x_2, x_3, \dots)$.
   - $l(x) = \lim_{n\to\infty} x_n$ for convergence sequences $x = (x_1, x_2, \dots)$,
   - l is nonnegative for nonnegative sequences.

   A linear functional of this type is called a Banach limit.

   Proof. Consider the shift operator S on $l^\infty(\mathbb{R})$ defined by $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Then S is linear. Let $Y := \{x - Sx : x \in l^\infty\}$. Then Y is a subspace of l^∞. If we write $u := (1, 1, 1, \dots)$ then we claim that $\operatorname{dist}(u, Y) = 1$. Indeed, since $0 \in Y$, $\operatorname{dist}(u, Y) \leq \|u\| = 1$. On the other hand, suppose $\|u - y\| < 1 - \varepsilon$ for some $\varepsilon > 0$ and $y \in Y$. Then we have, for some $x \in l^\infty$,

$$\sup\{|x_1 - x_2 - 1|, |x_2 - x_3 - 1|, |x_3 - x_4 - 1|, \dots\} < 1 - \varepsilon.$$

   Thus $x_1 - x_2 > \varepsilon$, $x_2 - x_3 > \varepsilon$, etc. This shows that the sequence $x_n \to -\infty$, which is a contradiction to the assumption that $x \in l^\infty$. Thus $\operatorname{dist}(u, Y) = 1$.

   Consider the closure of Y in l^∞, denoted by $\overline{Y}$. We can check $\overline{Y}$ is a proper subspace of l^∞, with $\operatorname{dist}(u, \overline{Y}) = 1$. By the result in Question 4, we can find a linear functional $l \in X^*$ such that $l(u) = 1$, $l \equiv 0$ on $\overline{Y}$, and $\|l\| = \operatorname{dist}(u, Y)^{-1} = 1$.

   It remains to check the required properties.

   - Since l is linear, it suffices to show $l(x_1 - x_2, x_2 - x_3, \dots) = 0$ for $x \in l^\infty$. But then $(x_1 - x_2, x_2 - x_3, \dots) \in Y \subseteq \overline{Y}$ on which l vanishes, and hence $l(x_1 - x_2, x_2 - x_3, \dots) = 0$.
   - If $\lim_{n\to\infty} x_n = x_\infty$, then given $\varepsilon > 0$, there is $N \in \mathbb{N}$ with $|x_n - x_\infty| < \varepsilon$ for all $n \geq N$. Denoting $y := (x_\infty, x_\infty, \dots)$ and using the shift invariant property repeatedly, we have

$$\begin{aligned} |l(x - y)| &= |l(x_1 - x_\infty, x_2 - x_\infty, \dots)| \\ &= |l(x_n - x_\infty, x_{n+1} - x_\infty, \dots)| \\ &\leq \|l\|\|(x_n - x_\infty, x_{n+1} - x_\infty, \dots)\|_\infty \\ &< \varepsilon. \end{aligned}$$

     Thus

$$|l(x) - x_\infty| \leq |l(x - y)| + |l(y) - x_\infty| < \varepsilon + |x_\infty l(u) - x_\infty| = \varepsilon.$$

     But since $\varepsilon > 0$ is arbitrary, we have $l(x) = x_\infty$.

- We write $l(x) = l(\|x\|u) - l(\|x\|u - x)$. Note

$$l(\|x\|u - x) = l(\|x\| - x_1, \|x\| - x_2, \dots) \le \|l\| \sup_n |\|x\| - x_n| \le \|x\|,$$

since $x_n \ge 0$. Hence

$$l(x) = l(\|x\|u) - l(\|x\|u - x) \ge l(\|x\|u) - \|x\| = \|x\|(l(u) - 1) = 0.$$

□

Idea for an alternative proof:

Proof. Define l on $Y :=$ the space of sequences such that the following limit exists.

$$l(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k.$$

Then l extends to l^∞ by the Hahn-Banach Theorem. To show l is shift invariant, note that $(x_1 - x_2, x_2 - x_3, \dots)$ is such that the above limit exists. □