MATH 421/510
Assignment 3

Suggested Solutions

February 2018

1. Let $\mathcal{H}$ be a Hilbert space.

   (a) Prove the polarization identity:

   $$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2).$$

   Proof. By direct calculation,

   $$\|x+y\|^2 - \|x-y\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle.$$

   Similarly,

   $$\|x+iy\|^2 - \|x-iy\|^2 = 2\langle x, iy\rangle - 2\langle y, ix\rangle = -2i\langle x, y\rangle + 2i\langle y, x\rangle.$$

   Addition gives

   $$\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle + 2\langle x, y\rangle - 2\langle y, x\rangle = 4\langle x, y\rangle.$$

   □

   (b) If there is another Hilbert space $\mathcal{H}'$, a linear map from $\mathcal{H}$ to $\mathcal{H}'$ is unitary if and only if it is isometric and surjective.

   Proof. We take the definition from the book that an operator is unitary if and only if it is invertible and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$.
   A unitary operator T is isometric and surjective by definition. On the other hand, assume it is isometric and surjective. We first show that T preserves the inner product:
   If the scalar field is $\mathbb{C}$, by the polarization identity,

   $$\begin{aligned}\langle Tx, Ty\rangle &= \frac{1}{4}(\|Tx+Ty\|^2 - \|Tx-Ty\|^2 + i\|Tx+iTy\|^2 - i\|Tx-iTy\|^2)\\ &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2) = \langle x, y\rangle,\end{aligned}$$

   where in the second equation we used the linearity of T and the assumption that T is isometric.

If the scalar field is $\mathbb{R}$, then we use the real version of the polarization identity:

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

The remaining computation is similar to the complex case.

It remains to show that T is injective. But

$$Tx = 0 \iff \langle Tx, Tx \rangle = 0 \iff \langle x, x \rangle = 0 \iff x = 0.$$

Hence T is injective. $\square$

2. If E is a subset of a Hilbert space $\mathcal{H}$, then $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing E.

Proof. For each subset A of $\mathcal{H}$, $A^\perp$ is always a subspace by definition. Also, it is closed, since if $y_n \to y$ in $\mathcal{H}$ where $\langle y_n, x \rangle = 0$, then $\langle y, x \rangle = 0$. With $A = E^\perp$, we have $(E^\perp)^\perp$ is a closed subspace. Moreover, it contains E by definition.

It remains to show minimality. let K be any closed subspace containing E, and we would like to show $(E^\perp)^\perp \subseteq K$. Suppose not. Then there is $x \in (E^\perp)^\perp$ with $x \notin K$. Since K is a proper closed subspace, by Question 4 of the last homework, there is $l \in \mathcal{H}^*$ such that $l(x) = 1$ and $l \equiv 0$ on K. By the Riesz-Fréchet theorem, there is $y \in \mathcal{H}$ with $l(x) = \langle x, y \rangle = 1 \neq 0$. Since $x \in (E^\perp)^\perp$, we have $y \notin E^\perp$. That means $\langle z, y \rangle \neq 0$ for some $z \in E$. But this is a contradiction since $l(z) = \langle z, y \rangle$ and $l \equiv 0$ on E. Therefore $(E^\perp)^\perp \subseteq K$. $\square$

3. Suppose $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$.

   (a) There is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$, called the adjoint of T, such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$.

   Proof. We first prove the existence. Fix $y \in \mathcal{H}$. Define the mapping $l_y(x) := \langle Tx, y \rangle$, which lies in $\mathcal{H}^*$. By the Riesz-Fréchet theorem, there is a unique $z \in \mathcal{H}$ with $l_y(x) = \langle x, z \rangle$ for all $x \in \mathcal{H}$, with $\|l_y\| = \|z\|$. We define $T^*y := z$ by the above relations. The uniqueness of z ensures that the mapping is well defined, and satisfies $\langle Tx, y \rangle = \langle x, T^*y \rangle$ by construction.

   One can check that T^* is linear and has operator norm 1. This shows the existence.

   To establish the uniqueness, it suffices to prove the following assertion: if T is a linear operator on a Hilbert space $\mathcal{H}$ and $\langle x, Ty \rangle = 0$ for all $x, y \in \mathcal{H}$, then $T \equiv 0$. Indeed, fix $y \in \mathcal{H}$, and taking $x = Ty$. Then we have $\langle Ty, Ty \rangle = 0$, whence $Ty = 0$. Hence $T \equiv 0$, so uniqueness is proved.

   $\square$

   (b) $\|T^*\| = \|T\|$, $\|T^*T\| = \|T\|^2$, $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$, $(ST)^* = T^*S^*$, and $T^{**} = T$.

Proof.
- We first show $T^{**} = T$. Indeed,

$$\langle T^{**}x, y\rangle = \overline{\langle y, T^{**}x\rangle} = \overline{\langle T^{*}y, x\rangle} = \langle x, T^{*}y\rangle = \langle Tx, y\rangle.$$

Since the above holds for all $x, y \in \mathcal{H}$, $T^{**} = T$.

- We then show $\|T^{*}\| = \|T\|$.
  - We first show $\|T^{*}\| \leq \|T\| < \infty$. Let $x \in \mathcal{H}$. If $T^{*}x = 0$, then we have nothing to prove. Otherwise, by the Cauchy-Schwarz inequality,

  $$\|T^{*}x\|^2 = \langle T^{*}x, T^{*}x\rangle = \langle x, TT^{*}x\rangle \leq \|x\|\|TT^{*}x\| \leq \|x\|\|T\|\|T^{*}x\|.$$

  Thus we have $\|T^{*}x\| \leq \|T\|\|x\|$. This shows that $\|T^{*}\| \leq \|T\|$.
  - Since $T^{**} = T$, we have $\|T\| = \|T^{**}\| \leq \|T^{*}\|$ by the previous direction. Hence $\|T^{*}\| = \|T\|$.
- We show $\|T^{*}T\| = \|T\|^2$.
  - On the one hand,

  $$\|T^{*}Tx\| \leq \|T^{*}\|\|T\|\|x\| = \|T\|^2\|x\|,$$

  where we have used $\|T^{*}\| = \|T\|$ in the last equality. Thus $\|T^{*}T\| \leq \|T\|^2 < \infty$.
  - On the other hand,

  $$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle = \langle x, T^{*}Tx\rangle \\ &\leq \|x\|\|T^{*}Tx\| \leq \|x\|\|T^{*}T\|\|x\|.\end{aligned}$$

  Hence $\|Tx\| \leq \|T^{*}T\|^{\frac{1}{2}}\|x\|$, so $\|T\| \leq \|T^{*}T\|^{\frac{1}{2}}$. Hence $\|T\|^2 \leq \|T^{*}T\|$.
- The other two equalities are direct.

□

(c) Let $\mathcal{R}$ and $\mathcal{N}$ denote the range and nullspace; then $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{*})$ and $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T^{*})}$.

Proof.
- $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{*})$:
  - $\mathcal{R}(T)^{\perp} \subseteq \mathcal{N}(T^{*})$: let $x \in \mathcal{R}(T)^{\perp}$. Then $\langle T^{*}x, y\rangle = \langle x, Ty\rangle = 0$ for all $y \in \mathcal{H}$. Taking $y = T^{*}x$ shows that $T^{*}x = 0$, that is, $x \in \mathcal{N}(T^{*})$.
  - $\mathcal{R}(T)^{\perp} \supseteq \mathcal{N}(T^{*})$: let $x \in \mathcal{N}(T^{*})$. Then $T^{*}x = 0$. For any $y \in \mathcal{H}$, $\langle x, Ty\rangle = \langle T^{*}x, y\rangle = 0$, which shows that $x \in \mathcal{R}(T)^{\perp}$.
- Applying the first part of the question to T^{*}, we have $\mathcal{R}(T^{*})^{\perp} = \mathcal{N}(T^{**}) = \mathcal{N}(T)$, so $(\mathcal{R}(T^{*})^{\perp})^{\perp} = \mathcal{N}(T)^{\perp}$. But by Question 2 in this homework, $(\mathcal{R}(T^{*})^{\perp})^{\perp}$ is the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{R}(T^{*})$. Since $\mathcal{R}(T^{*})$ is already a subspace of $\mathcal{H}$, the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{R}(T^{*})$ is $\overline{\mathcal{R}(T^{*})}$. Hence $\mathcal{N}(T)^{\perp} = \overline{\mathcal{R}(T^{*})}$.

□

(d) T is unitary if and only if T is invertible and $T^{-1} = T^{*}$.

Proof. As in Question 1 (b), we take the definition that an operator is unitary if and only if it is invertible and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$.

If T is unitary, then T is invertible by definition. To show that $T^{-1} = T^*$, we note that

$$\langle Tx, y\rangle = \langle Tx, TT^{-1}y\rangle = \langle x, T^{-1}y\rangle,$$

for all $x, y \in \mathcal{H}$, since T is unitary. This is to say that T^{-1} is an adjoint of T. By uniqueness of the adjoint operator, we have $T^{-1} = T^*$.

On the other hand, suppose T is invertible and $T^{-1} = T^*$. To show T is unitary, it suffices to show it preserves the inner product. But we easily compute

$$\langle Tx, Ty\rangle = \langle x, T^*Ty\rangle = \langle x, T^{-1}Ty\rangle = \langle x, y\rangle,$$

for all $x, y \in \mathcal{H}$. hence T is unitary.

□

4. Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$, and for $x \in \mathcal{H}$ let Px be the element of $\mathcal{M}$ such that $x - Px \in \mathcal{M}^\perp$ as in Theorem 5.24.

   (a) $P \in L(\mathcal{H}, \mathcal{H})$, $P^* = P$, $P^2 = P$, $\mathcal{R}(P) = \mathcal{M}$, and $\mathcal{N}(P) = \mathcal{M}^\perp$. P is called the orthogonal projection onto $\mathcal{M}$.

   Proof. i. P is linear: Theorem 5.24 states that each $x \in \mathcal{H}$ can be uniquely decomposed into $x = y + z$, where $Px := y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. Using this and the fact that $\mathcal{M}$, $\mathcal{M}^\perp$ are subspaces, we can prove linearity.
   P is bounded: Theorem 5.24 also states that Px is perpendicular to $x - Px$. By the Pythagorean Theorem,

   $$\|Px\|^2 = \|x\|^2 - \|x - Px\|^2 \le \|x\|^2,$$

   which shows that $\|P\| \le 1$. Hence $P \in L(\mathcal{H}, \mathcal{H})$.

   ii. By uniqueness of the adjoint operator, it suffices to show that for all $x, x' \in \mathcal{H}$, we have

   $$\langle Px, x'\rangle = \langle x, Px'\rangle.$$

   Decompose $x = y + z$, $x' = y' + z'$ as in Theorem 5.24. This can be proved using the fact that $y, y' \in \mathcal{M}$ and $z, z' \in \mathcal{M}^\perp$.

   iii. Note that for each $y \in \mathcal{M}$, the orthogonal decomposition in Theorem 5.24 is $y = y + 0$, so $Py = y$. (This implies that $\|P\| = 1$ unless $\mathcal{M} = \{0\}$). Applying this to $y = Px \in \mathcal{M}$, we have $P^2x = P(Px) = Px$ for all $x \in \mathcal{H}$.

   iv. We have $\mathcal{R}(P) \subseteq \mathcal{M}$ by definition. On the other hand, given any $x \in \mathcal{M}$, $Px = x$ implies that $x \in \mathcal{R}(P)$.

   v. Since $P = P^*$, $\mathcal{N}(P) = \mathcal{N}(P^*)$. By Question 3(c), $\mathcal{N}(P^*) = \mathcal{R}(P)^\perp$. But $\mathcal{R}(P) = \mathcal{M}$, so $\mathcal{R}(P)^\perp = \mathcal{M}^\perp$. Hence $\mathcal{N}(P) = \mathcal{M}^\perp$.

   □

   (b) Conversely, suppose that $P \in L(\mathcal{H}, \mathcal{H})$ satisfies $P^2 = P^* = P$. Then $\mathcal{R}(P)$ is closed and P is the orthogonal projection onto $\mathcal{R}(P)$.

Proof. By Theorem 5.24, it suffices to show that $\mathcal{M} := \mathcal{R}(P)$ is closed, and then show that $Px = x$ for all $x \in \mathcal{M}$ and $Px = 0$ for all $x \in \mathcal{M}^\perp$.

To show that $\mathcal{R}(P)$ is closed, note that $P^2 = P$ implies that $\mathcal{R}(P) = \mathcal{N}(P-I)$. Recall that the nullspace of a bounded linear operator is closed. Using this fact to the bounded linear operator $P - I$, we have $\mathcal{R}(P)$ is closed.

Next, let $x \in \mathcal{M} = \mathcal{R}(P)$. Then $x = Py$ for some $y \in \mathcal{H}$. Since $P^2 = P$, $Px = P^2y = Py = x$.

Lastly, let $x \in \mathcal{M}^\perp = \mathcal{R}(P)^\perp$. By Question 3(c), $\mathcal{R}(P)^\perp = \mathcal{N}(P^*)$. But $P^* = P$, so $\mathcal{N}(P^*) = \mathcal{N}(P)$. Hence $Px = 0$. This completes the proof. □

(c) If $\{u_\alpha\}$ is an orthonormal basis for $\mathcal{M}$, then $Px = \sum \langle x, u_\alpha \rangle u_\alpha$.

Proof. By properties of P, we have $\langle x, u \rangle = \langle Px, u \rangle$ for all $u \in \mathcal{M}$.

Let $x \in \mathcal{H}$, then $Px \in \mathcal{M}$. By definition of the orthonormal basis, there are c_α, where at most countably many are nonzero, such that

$$Px = \sum_\alpha c_\alpha u_\alpha.$$

Moreover, the sum on the right is absolutely convergent in $\mathcal{H}$.

Now fix β. We have, by the continuity of the inner product,

$$\langle Px, u_\beta \rangle = \Big\langle \sum_\alpha c_\alpha u_\alpha, u_\beta \Big\rangle = \sum_\alpha c_\alpha \langle u_\alpha, u_\beta \rangle.$$

Since $\{u_\alpha\}$ is orthonormal, $\langle u_\alpha, u_\beta \rangle = 0$ or 1 according as $\alpha \neq \beta$ or $\alpha = \beta$. Hence $\langle Px, u_\beta \rangle = c_\beta$ for all β. This proves the claim. □

5. In this exercise the measure defining the L^2 spaces is the Lebesgue measure.

(a) $C([0,1])$ is dense in $L^2([0,1])$. (Adapt the proof of Theorem 2.26).

Proof. Let $f \in L^2([0,1])$ and let $\varepsilon > 0$. Then there is a large N such that $\left\| f 1_{(|f|>N)} \right\|_2 < \varepsilon/2$. (This can be proved using the dominated convergence theorem). Define $g := f 1_{(|f| \leq N)}$. By Lusin's theorem (Page 64 in Folland), there is a compact $E \subseteq [0,1]$ such that $g|_E$ is continuous and $[0,1] \backslash E$ has measure less than $\varepsilon^2/(16N^2)$. Furthermore, by Tietze extension theorem (Page 122 in Folland), $g|_E$ can be extended to $h : [0,1] \to \mathbb{C}$ such that h is continuous on $[0,1]$, with $\|h\|_\infty \leq \|g\|_\infty \leq N$. This h is the required continuous function. Indeed,

$$\int_0^1 |h-g|^2 = \int_{[0,1]\backslash E} |h-g|^2 \leq \int_{[0,1]\backslash E} 4N^2 < \frac{\varepsilon^2}{4}.$$

Thus $\|h-g\|_2 < \frac{\varepsilon}{2}$. Since $\|f-g\|_2 < \frac{\varepsilon}{2}$, the triangle inequality shows that $\|f-h\|_2 < \varepsilon$. □

Remark from Marking:

Alternative answer: We have a fact that if $f \in L^2([0,1])$, then the Fourier series of f converges to f in $L^2([0,1])$. Since any partial sum of the Fourier series is continuous, we are done.

Most standard answer: Approximate f by simple functions, then approximate indicator functions of measurable sets by a linear combination of indicator functions of intervals, and lastly, approximate linear combinations of indicator functions of intervals by continuous functions.

(b) The set of polynomials is dense in $L^2([0,1])$.

Proof. Let $f \in L^2([0,1])$ and let $\varepsilon > 0$. By Part (a), there is $h \in C([0,1])$ with $\|f-h\|_2 < \varepsilon/2$. Next, by Weierstrass approximation theorem, there is a polynomial P such that $\|P-h\|_\infty < \varepsilon/2$. But then Hölder's inequality shows that

$$\|P-h\|_2 \leq \|P-h\|_\infty \|1\|_2 = \|P-h\|_\infty < \varepsilon/2.$$

Again, the triangle inequality shows that $\|f-P\|_2 < \varepsilon$. □

(c) $L^2([0,1])$ is separable.

Proof. By Part (b), the set of all polynomials on $[0,1]$ is dense in $L^2([0,1])$. Furthermore, any polynomial with complex coefficients can be uniformly (and hence in L^2 by Hölder's inequality) approximated by polynomials with coefficients in $\mathbb{Q}^2$. Hence the set of the latter is a countable dense subset of $L^2([0,1])$. □

(d) $L^2(\mathbb{R})$ is separable. (Use Exercise 60.)

Proof. Use Exercise 60 and the decomposition $\mathbb{R} = [n, n+1)$, and note that $L^2([n, n+1))$ is separable by a trivial modification of Part (c). □

(e) $L^2(\mathbb{R}^n)$ is separable. (Use Exercise 61.)

Proof. We prove it by induction. The case $n = 1$ is the statement of Part (d). Suppose $L^2(\mathbb{R}^n)$ is separable, $n \geq 1$. By Proposition 5.29 in Folland, a Hilbert space $\mathcal{H}$ is separable iff it has a countable orthonormal basis, in which case every orthonormal basis for $\mathcal{H}$ is countable. This proposition, together with Exercise 61, show that $L^2(\mathbb{R}^{n+1})$ is separable. □