MATH 421/510
Assignment 4

Suggested Solutions

March 2018

1. Let $Y = L^1(\mu)$ where μ is the counting measure on $\mathbb{N}$, and let $X = \{f \in Y : \sum_{n=1}^{\infty} n|f(n)| < \infty\}$, equipped with L^1-norm.

   (a) X is a proper dense subspace of Y; hence X is not complete.

   Proof. • It is direct to check that X is a subspace of Y.

   • $X \subsetneqq Y$, since $f(n) := n^{-2} \in Y$ but not in X.

   • X is dense in Y. Too see this, let $x \in Y$ and $\varepsilon > 0$. Then there is N such that $\sum_{n=N}^{\infty} |f(n)| < \varepsilon$. But the truncated sequence $g(n) := f(n)1_{(n<N)}$ clearly lies in X and satisfies $\sum_{n=1}^{\infty} |f(n) - g(n)| < \varepsilon$.

   □

   (b) Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.

   Proof. • By definition, T is a closed linear operator (not a closed map!!), if $f_m \to f$ in X and $Tf_m \to g$ in Y implies that $g = Tf$. In our case, we are to show

   $$g(n) = nf(n) \quad \forall n \in \mathbb{N},$$

   given that

   $$\lim_{m\to\infty} \sum_{n=1}^{\infty} |f_m(n) - f(n)| = 0, \tag{1}$$

   $$\lim_{m\to\infty} \sum_{n=1}^{\infty} |nf_m(n) - g(n)| = 0, \tag{2}$$

   In particular, for any $n \in \mathbb{N}$, (1) implies that $\lim_{m\to\infty} f_m(n) = f(n)$, and (2) implies that $\lim_{m\to\infty} nf_m(n) = g(n)$. Combining these two gives $g(n) = nf(n)$, as desired.

   Comment. Many of you proved the statement that T is a topologically closed map. It is an exercise to show that this is stronger than T being a closed linear operator.

Reference: https://math.stackexchange.com/questions/2205068/example-of-a-linear-operator-whose-graph-is-not-closed-but-it-takes-a-closed-set?rq=1

- Consider $f_m(n) := e_m$ for $m \in \mathbb{N}$, where $\{e_m\}_{m=1}^{\infty}$ is the canonical basis for $L^1(\mu)$. Then $\|Tf_m\|_1 = m$, so

$$\sup_{f\in X, \|f\|_1=1} \frac{\|Tf_m\|_1}{\|f_m\|_1} \geq \frac{m}{1} = m.$$

Since m can arbitrarily large, T is unbounded.

□

(c) Let $S = T^{-1}$. Then $S : Y \to X$ is bounded and surjective but not open.

Proof.
- Clearly, S is well defined by $Sf(n) = f(n)/n$. It is bounded since

$$\|Sf\|_1 = \sum_{n=1}^{\infty} \frac{|f(n)|}{n} \leq \sum_{n=1}^{\infty} |f(n)| = \|f\|_1.$$

- S is surjective, since given any $f \in X$, we have $Tf \in Y$ and $S(Tf) = f$ by definition.
- S is open if and only if $S^{-1} = T$ is continuous if and only if T is bounded since T is linear. But T is unbounded, so S is not open.

□

2. Let $Y = C[0,1]$ and $X = C^1[0,1]$, both equipped with the uniform norm.

(a) X is not complete.

Proof. By the Weierstrass approximation theorem, the space of all polynomials P is dense in Y under the sup-norm. Since $P \subseteq X$, that means X is also dense in Y. If X is complete, then $X = Y$, which is absurd. Thus X cannot be complete. □

(b) The map $(d/dx) : X \to Y$ is closed but not bounded.

Proof.
- To show the map is closed, let $f_n \to f$ in X, $f_n' \to g$ in Y, and our goal is to show that $g = f'$. This is proved in Problem 3(b) of Homework 1.
- The map is not bounded, as can be seen from the examples $x^n \mapsto nx^{n-1}$, $n \in \mathbb{N}$.

□

3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space X such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If X is complete with respect to both norms, then the norms are equivalent.

Proof. Define $I : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ to be the identity map. This maps is clearly linear and surjective, and $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are both complete by assumption. Moreover, $\|I\|_{\text{op}} \leq 1$. By the open mapping theorem, I is open, which means that $I^{-1} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous, and hence bounded. Thus there is C with $\|\cdot\|_2 \leq C\|\cdot\|_1$, so the norms are equivalent. □

4. There is no slowest rate of decay of the terms of an absolutely convergence series; that is, there is no sequence $\{a_n\}$ of positive numbers such that $\sum a_n|c_n| < \infty$ if and only if $\{c_n\}$ is bounded.

Proof. Suppose there is such sequence $\{a_n\}$. Define $T : B(\mathbb{N}) \to L^1(\mu)$ by $Tf(n) = a_n f(n)$, where $B(\mathbb{N})$ is the space of all bounded sequences endowed with the sup-norm. The assumption is to say that T is well defined and invertible, with $T^{-1}f(n) = a_n^{-1}f(n)$.

The mapping T is bounded, which we now show. By definition of $\{a_n\}$, if we take $c_n = e := (1, 1, 1, \dots) \in B(\mathbb{N})$, then we get $\sum a_n < \infty$. Thus

$$\|Tf\|_1 = \sum_{n=1}^{\infty} a_n|f(n)| \leq \|f\|_\infty \sum_{n=1}^{\infty} a_n,$$

so T is bounded. By the open mapping theorem, T is open. Therefore T is a homeomorphism between the spaces $B(\mathbb{N})$ and $L^1(\mu)$.

Consider S, the set of f such that $f(n) = 0$ for all but finitely many n. S is dense in L^1, which is proved in Q1 (a). But S is not dense in $B(\mathbb{N})$. For, consider $e \in B(\mathbb{N})$. If $h \in S$ is any finite sequence, then $\|g - h\|_\infty \geq 1$.

But T is a homeomorphism between $B(\mathbb{N})$ and $L^1(\mu)$, and S is dense in $L^1(\mu)$, so $T^{-1}(S)$ is dense in $B(\mathbb{N})$. But $T^{-1}(S) \subseteq S$, so S is dense in $B(\mathbb{N})$, which is a contradiction. Therefore, such positive sequence $\{a_n\}$ does not exist. □

5. Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

Proof. Since X and Y are Banach spaces, to show that T is bounded, it is equivalent to showing that T is a closed linear operator.

Let $x_n \to x$ in X and $Tx_n \to y$ in Y. To show that $Tx = y$, we claim that it is equivalent to showing that $f(Tx) = f(y)$ for all $f \in Y^*$, which is exactly our assumption. Indeed, by linearity, if $Tx - y \neq 0$, then by a corollary of the Hahn-Banach theorem (Q4 of Homework 2), there is $f \in Y^*$ such that $f(Tx - y) = 1$, which is a contradiction. Hence $Tx = y$ and T is closed. □

6. Let X and Y be Banach spaces, and let T_n be a sequence in $L(X, Y)$ such that $\lim_n T_n x$ exists for every $x \in X$. Let $Tx = \lim_n T_n x$; then $T \in L(X, Y)$.

Proof. Let $x \in X$. Since $Tx = \lim_n T_n x$ exists, in particular, $\{T_n x\}$ is bounded in n. Since X is a Banach space, the uniform boundedness principle implies that $\sup_n \|T_n\|_{\text{op}} \leq M < \infty$. Thus

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \sup_n \|T_n\|_{\text{op}} \|x\| \leq M\|x\|.$$

Since T is obviously linear, $T \in L(X, Y)$. □

7. Let X and Y be Banach spaces and $\{T_{jk} : j, k \in \mathbb{N}\} \subseteq L(X, Y)$. Suppose that for each k there exists $x \in X$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$. Then there is an x such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$ for all k.

Proof. We prove it by contradiction. Suppose there is no such x. Then for all x, there is k_x such that the sequence $\sup\{\|T_{jk_x}x\| : j \in \mathbb{N}\} < \infty$. Thus we can write

$$X = \bigcup_{k=1}^{\infty} \left\{ x : \sup_j \|T_{jk}x\| < \infty \right\} := \bigcup_{k=1}^{\infty} E_k.$$

Denote $E_{k,n} := \{x : \sup_j \|T_{jk}x\| \leq n\}$, and hence $X = \cup_{k=1}^{\infty} \cup_{n=1}^{\infty} E_{k,n}$.

- Each $E_{k,n}$ is closed: given $x_m \subseteq E_{k,n}$ with $x_m \to x$, then for all j we have

$$\|T_{jk}x\| = \lim_m \|T_{jk}x_m\| \leq n,$$

  since T_{jk} is continuous and $x_m \in E_{k,n}$. Hence $x \in E_{k,n}$.
- Each $E_{k,n}$ is nowhere dense. To see this, note first it is easy to check that E_k is a subspace of X; moreover, $E_k \subsetneqq X$ by the assumption that there is $x \in X$ such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$. Hence E_k is a proper subspace of X, so E_k is nowhere dense. As a subset of E_k, $E_{k,n}$ is also nowhere dense.

Since X is a Banach space, we have reached a contradiction to the Baire category theorem. Hence our assumption is false, that is, there is an x such that $\sup\{\|T_{jk}x\| : j \in \mathbb{N}\} = \infty$ for all k. □