MATH 421/510
Assignment 5

Suggested Solutions

April 2018

1. Given any point $t_0 \in [0, 2\pi]$, show using the uniform boundedness principle that there exists a continuous 2π-periodic function whose Fourier series diverges at t_0. We sketched a proof of this result in class. Fill in the details.

   Proof. Consider the N-th Dirichlet kernel

   $$D_N(t) := \sum_{n=-N}^{N} e^{int}.$$

   It is a fact that the partial sum sequence $S_N f(t) = (D_N * f)(t)$, where the integral defining the convolution is normalized by a factor $1/2\pi$:

   $$S_N f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t-s) f(s) ds.$$

   We show this in several steps, using contradiction. Suppose for all $f \in C[-\pi, \pi]$, we have $S_N f(t_0) \to f(t_0)$. Then:

   (a) The mapping $l_N : f \mapsto S_N f(t_0)$ is linear and bounded from $C[-\pi, \pi] := (C[-\pi, \pi], \|\cdot\|_\infty)$ to $\mathbb{C}$, with a $\sup_N \|l_N\| \leq C < \infty$. This is a result of the uniform boundedness principle.

   (b) We show that this implies that S_N is bounded from $C[-\pi, \pi]$ to $C[-\pi, \pi]$, with the bound independent of N. Indeed, given $f \in C[-\pi, \pi]$, suppose $|S_N f|$ attains its maximum at t_1. Consider the translated function $g(t) := f(t + t_1 - t_0)$, which has $\|g\|_\infty = \|f\|_\infty$ and $S_N g(t_0) = S_N f(t_1)$. Hence

   $$\|S_N f\|_\infty = |S_N f(t_1)| = |S_N g(t_0)| \leq C\|g\|_\infty = C\|f\|_\infty.$$

   (c) We state a special case of the Young's convolution theorem:

   Theorem 1. Let (X, μ) be a measure space, and g be a measurable function. The convolution operator $T : f \mapsto f * g$ is bounded from L^∞ to L^∞ if and only if $g \in L^1$. Moreover, $\|T\|_{L^\infty \to L^\infty} = \|g\|_{L_1}$.

   Now in our situation, $\sup_N \|S_N\| < \infty$ implies that $\sup_N \|D_N\|_1 < \infty$.

(d) Lastly, we show the above cannot happen. Direct computation shows that

$$D_N(x) = \frac{\sin\left(N + \frac{1}{2}\right)x}{\sin\left(\frac{1}{2}x\right)}.$$

By considering the integral over $|x| \in [k\pi/(N+\frac{1}{2}), (k+1)\pi/(N+\frac{1}{2})]$ for each k, we see $\|D_N\|_1$ is bounded below by a constant times the first N terms of the harmonic series. Letting $N \to \infty$, we have $\|D_N\|_1 \to \infty$, contradiction to the conclusion above.

$\square$

2. In class, we introduced the concept of a locally convex space, whose topology is generated by a family of seminorms. When is such a topology equivalent to a metric topology? A norm topology?

Note: If X is locally convex, it separates points by definition taught in class.

(a) We claim such a topology is a metric topology if and only if it is generated by a countable family of seminorms.

Proof. ("$\Longleftarrow$") Let $\{p_i\}_{i=1}^{\infty}$ be the countable family of seminorms that generates a topology on X. Then we define a metric by

$$d(x,y) := \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x-y)}{1+p_i(x-y)}.$$

It is direct to check that d is a metric. Indeed, $d(x,y) \geq 0$, and if $d(x,y) = 0$, then $p_i(x-y) = 0$ for all i. Since $\{p_i\}$ separates points, we have $x = y$.

Symmetry is trivial. For the triangle inequality, refer to the following question: https://math.stackexchange.com/questions/309198/if-d-is-a-metric-then-d-1d-is-also-a-metric.

It remains to show d generates the same topology as $\{p_i\}_{i=1}^{\infty}$ does. By translation invariance, it suffice to consider their neighbourhood bases at 0:

$$B_d(\varepsilon) := \{x \in X : d(x,0) < \varepsilon\},$$

$$\bigcap_{i=1}^{n} B_i(\varepsilon_i) := \{x \in X : p_i(x) < \varepsilon_i \quad \forall 1 \leq i \leq n\}.$$

- Given $\varepsilon > 0$, take N such that $\sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2$. Take $\varepsilon_i := \varepsilon/2$ for all $1 \leq i \leq N$. Thus if $p_i(x) < \varepsilon/2$ for all $1 \leq i \leq N$, we have

$$d(x,0) \leq \sum_{i=1}^{N} 2^{-i}\frac{\varepsilon}{2} + \sum_{i=N+1}^{\infty} 2^{-i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

  This implies that

$$\bigcap_{i=1}^{N} B_i(\varepsilon_i) \subseteq B_d(\varepsilon).$$

- On the other hand, given $\varepsilon_i > 0$ for $i = 1, 2, \ldots, n$, then $d(x, 0) < \varepsilon := \min\{\varepsilon_i : 1 = 1, 2, \ldots, n\}$ implies that $p_i(x) < \varepsilon_i$. Hence

$$B_d(\varepsilon) \subseteq \bigcap_{i=1}^{n} B_i(\varepsilon_i).$$

  Therefore they generate the same topology.

(" $\Longrightarrow$ ") Note that $\{B_d(1/n)\}_{n=1}^{\infty}$ forms a neighbourhood base at 0. For each n, there is $B_{i,n}(\varepsilon_{i,n}), i = 1, 2, \ldots, K_n$ such that

$$B_d\left(0, \frac{1}{n}\right) \supseteq \bigcap_{i=1}^{K_n} B_{i,n}(\varepsilon_{i,n}).$$

Relabel the countable collection $\{p_{i,n} : 1 \leq i \leq K_n, n \in \mathbb{N}\}$ as $\{p_j\}_{j=1}^{\infty}$. Then $\{p_j\}_{j=1}^{\infty}$ generates the metric topology d.

□

(b) We claim such a topology is a norm topology if and only if it is generated by a finite collection of seminorms.

Proof. (" $\Longleftarrow$ ") Let $\mathcal{P} := \{p_i\}_{i=1}^{N}$ be the finite collection of seminorms that generates a topology on X. Then we define a norm by

$$\|x\| := \max\{p_i(x), i = 1, 2, \ldots, N\}.$$

It is direct to check that $\|\cdot\|$ is a norm. Indeed, $\|x\| \geq 0$, and if $\|x\| = 0$, then $p_i(x) = 0$ for all i. Since $\{p_i\}$ separates points, we have $x = 0$.
Homogeneity and the triangle inequality follows from the corresponding properties of the seminorms.
It remains to show $\|\cdot\|$ generates the same topology as $\{p_i\}_{i=1}^{N}$ does. But this is similar and easier than the countable case.
(" $\Longrightarrow$ ") This is trivial. □

3. Let (X, Ω, μ) be a σ-finite measure space, $1 \leq p < \infty$. Suppose that $K : X \times X \to \mathbb{F}$ is an $\Omega \times \Omega$-measurable function such that for $f \in L^p(\mu)$ and almost every $x \in X$, the function $K(x, \cdot)f(\cdot) \in L^1(\mu)$ and

$$\mathcal{K}f(x) = \int K(x, y)f(y)d\mu(y)$$

defines an element $\mathcal{K}f \in L^p(\mu)$. Show that $\mathcal{K}$ is a bounded operator on $L^p(\mu)$.

Proof. We first prove a lemma:

Lemma 1. Let (X, Ω, μ) be a measure space, and f be a measurable function. Suppose $\int_X fg$ converges absolutely for every $f \in L^p$, $1 \leq p \leq \infty$. Then $g \in L^{p'}$, where p' is the dual exponent of p.

Proof of the Lemma. Suppose, towards contradiction, that $g \notin L^{p'}$. By duality, this is to say that there is a sequence $f_n \in L^p$ with $\|f_n\|_p = 1$ such that $\left|\int_X f_n g\right| > 4^n$. In particular, $\int_X |f_n g| > 4^n$.

Now we define $f := \sum_{n=1}^\infty 2^{-n}|f_n|$. We have $\|f\|_p \le \sum_{n=1}^\infty 2^{-n}\|f_n\|_p = 1$, by the triangle inequality. However, we see that

$$\int_X |fg| \ge \sum_{n=1}^\infty 2^{-n} \int_X |f_n g| > \sum_{n=1}^\infty 2^{-n} 4^n = \infty,$$

so $fg \notin L^1$, a contradiction. Hence $g \in L^{p'}$. □

By the lemma, $K(x,\cdot) \in L^{p'}$ for a.e. $x \in X$. By Hölder's inequality, $f \mapsto \mathcal{K}f(x)$ is a bounded linear functional on L^p for a.e. $x \in X$.

We will use the closed graph theorem to show that $\mathcal{K}$ is bounded on L^p. Let $f_n \to f$ in L^p, $\mathcal{K}f_n \to g$ in L^p. Since $f \mapsto \mathcal{K}f(x)$ is continuous on L^p for a.e. x, $\mathcal{K}f_n(x) \to \mathcal{K}f(x)$ a.e. By uniqueness of limits, we have $g = \mathcal{K}f(x)$, which completes the proof.

□

4. (a) Show that the weak topology on X is the weakest topology for which all $l \in X^*$ is continuous.

Proof. We take the definition of weak topology on X as the topology generated by the seminorms $p(x) := |l(x)|$ over $l \in X^*$.

Recall that a linear functional $l : X \to \mathbb{F}$ is continuous if and only if there exists finitely many seminorms p_i, $1 \le i \le n$, and a constant C such that for all $x \in X$,

$$|l(x)| \le C \sum_{i=1}^n p_i(x).$$

Now we take $C = 1$ and a single $p_1 = |l|$ to finish the proof.

On the other hand, given any topology on X such that each $l \in X^*$ is continuous. Since taking modulus on the scalar field is continuous, we see that each $x \mapsto p(x) = |l(x)|$ is continuous. Hence the weak topology is weaker than any topology such that each $l \in X^*$ is continuous. Lastly, by taking intersection of all such topologies, we see that the weak topology on X is unique, so it is indeed the weakest topology such that each $l \in X^*$ is continuous. □

(b) Show that the weak-star topology is the smallest topology on X^* such that for each $x \in X$, the map $l \mapsto l(x)$ is continuous.

Proof. We take the definition of weak-star topology on X^* as the topology generated by the seminorms $q_x(l) := |l(x)|$ over $x \in X$.

Recall that a linear functional $q : X^* \to \mathbb{F}$ is continuous if and only if there exists finitely many seminorms q_{x_i}, $1 \le i \le n$, and a constant C such that for all $l \in X^*$,

$$|q(l)| \le C \sum_{i=1}^n q_{x_i}(l) = C \sum_{i=1}^n |l(x_i)|.$$

Now for each $x \in X$, $q(l) = q_x(l)$ is the mapping $l \mapsto |l(x)|$. We take $C = 1$ and a single $x_1 = x$ to finish the proof.

On the other hand, given any topology on X^* such that for each $x \in X$, $l \mapsto l(x)$ is continuous. Since taking modulus on the scalar field is continuous, we see that each $l \mapsto q_x(l) = |l(x)|$ is continuous. Hence the weak star topology is weaker than any topology with the aforesaid property. Uniqueness is similar as the above. □

5. (a) If $\mathbb{H}$ is a Hilbert space and $\{h_n\} \subseteq \mathbb{H}$ is a sequence such that $h_n \to h$ weakly and $\|h_n\| \to \|h\|$, then show that $h_n \to h$ strongly.

Proof. Since $\mathbb{H}$ is self dual, $h_n \to h$ weakly if and only if for all $g \in \mathbb{H}$, we have $\langle h_n, g\rangle \to \langle h, g\rangle$. Taking $g = h$, we have $\langle h_n, h\rangle \to \langle h, h\rangle$.

By assumption, $\langle h_n, h_n\rangle = \|h_n\|^2 =\to \|h\|^2 = \langle h, h\rangle$. Therefore

$$\begin{aligned}\langle h_n - h, h_n - h\rangle &= \langle h_n, h_n\rangle - \langle h_n, h\rangle - \langle h, h_n\rangle + \langle h, h\rangle \\ &\to \langle h, h\rangle - \langle h, h\rangle - \langle h, h\rangle + \langle h, h\rangle \\ &= 0.\end{aligned}$$

□

(b) Prove the same statement for the Lebesgue spaces $L^p(\mu)$, $1 < p < \infty$.

Proof. We will use the fact that $L^p(\mu)$ is uniformly convex for $1 < p < \infty$, that is, for each $0 < \varepsilon < 1$, there is $\delta > 0$ such that for all $\|f\|_p = 1 = \|g\|_p$, $\|f - g\|_p > \varepsilon$ implies that $\|(f+g)/2\|_p < 1 - \delta$. This is a direct result of the Clarkson's inequalities (an elementary calculation with $\varepsilon - \delta$ involved):

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \le \frac{1}{2}(\|f\|_p^p + \|g\|_p^p), \qquad \text{if} \qquad 2 \le p < \infty; \tag{1}$$

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \le \left(\frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p\right)^{\frac{p'}{p}}, \qquad \text{if} \qquad 1 < p < 2; \tag{2}$$

where $1/p + 1/p' = 1$.

For those who are interested in the proof of Clarkson's inequalities, you can find one on Page 15 in the following lecture notes: http://www.math.cuhk.edu.hk./course_builder/1718/math5011/MATH5011_Chapter_4.2017%20.pdf

We still need another tool, namely, Fatou's lemma on weakly convergent sequences:

Lemma 2. Let $x_n \rightharpoonup x$ in a normed space X. Then $\|x\| \le \liminf_{n\to\infty}\|x_n\|$.

Proof of the Lemma. Using duality, we have $\|x\| = \sup_{\|f\|_{X^*}=1}|f(x)|$. Now let $f \in X^*$ with $\|f\|_{X^*} = 1$. We have

$$|f(x)| = |\lim_{n\to\infty} f(x_n)| = \lim_{n\to\infty}|f(x_n)| \le \liminf_{n\to\infty}\|f\|_{X^*}\|x_n\| = \liminf_{n\to\infty}\|x_n\|.$$

Since $f \in X^*$, $\|f\|_{X^*} = 1$ is arbitrary, we have $\|x\| \le \liminf_{n\to\infty}\|x_n\|$. □

We now come to the proof of the analogous statement as above. Since $f_n \rightharpoonup f$, $(f_n + f)/2 \rightharpoonup f$. By Fatou's lemma on weakly convergent sequences, we have

$$\|f\|_p \leq \liminf_{n\to\infty} \left\| \frac{f_n + f}{2} \right\|_p .$$

On the other hand, we also have

$$\left\| \frac{f_n + f}{2} \right\|_p \leq \frac{1}{2}\|f_n\|_p + \frac{1}{2}\|f\|_p \to \|f\|_p,$$

which follows from the assumption that $\|f_n\|_p \to \|f\|_p$. This forces that all the above inequalites should be equalities, whence we have

$$\lim_{n\to\infty} \left\| \frac{f_n + f}{2} \right\|_p = \|f\|_p.$$

Lastly, either using the uniform convexity, or just plugging $g = f_n$ in the Clarkson's inequality which is simpler in this case, and taking limits $n \to \infty$, we have $\|f - f_n\|_p \to 0$. □

6. Suppose that X is an infinite-dimensional normed space. Find the weak closure of the unit sphere.

Proof. (*Credit to Jeffrey Dawson for this solution*)

We claim that the weak closure of the unit sphere S is the closed unit ball $B := \{x \in X : \|x\| \leq 1\}$. (Remark: for a normed space, the closed unit ball is equal to the closure of the (open) unit ball, which is not true for a general metric space.)

We claim that

$$B = \bigcap_{\|l\|=1} \{x : |l(x)| \leq 1\}.$$

Indeed, if $\|x\| \leq 1$, then $|l(x)| \leq 1$ whenever $\|l\| = 1$; on the other hand, if $\|x\| > 1$, then by the Hahn-Banach theorem, there is $l \in X^*$ such that $\|l\| = 1$ and $l(x) = \|x\| > 1$. This proves the claim above.

Since each $\{x : |l(x)| \leq 1\}$ is weakly closed, so is any intersection over $\|l\| = 1$. Hence B is a weakly closed set containing S, so B contains the weak closure of S.

On the other hand, let $x_0 \in B$; we want to show that x_0 is in the weak closure of S. To do this, let G be a weakly open set containing x_0, and without loss of generality, assume G is a basic weakly open neighbourhood of x_0, that is, there are $l_i \in X^*$, $\delta_i > 0$, $1 \leq i \leq n$, such that

$$G = \bigcap_{i=1}^{n} \{x : |l_i(x - x_0)| < \delta_i\}.$$

Now we take $0 \neq y \in \cap_{i=1}^{n} \mathrm{Ker}(l_i)$; this is possible since the right hand side has codimension $n < \infty$ while X is infinite-dimensional. The functions $\lambda \mapsto \|\lambda y + x_0\|$ is a continuous function which sends 0 to $\|x_0\| \leq 1$ and tends to ∞ as $\lambda \to \infty$.

By the intermediate value theorem, there is $\lambda \geq 0$ such that $\|\lambda y + x_0\| = 1$. Let $x = \lambda y + x_0$, then $\|x\| = 1$ and $l_i(x - x_0) = l_i(\lambda y) = 0$ for all i, so $x \in G$, and thus $G \cap S \neq \varnothing$. Since G is arbitrary, x_0 is in the weak closure of S.

Combining two sides finishes the proof. $\square$