Math 421/510 Quiz 2 Solution

1. For an infinite-dimensional normed vector space X of your choice, find two closed, bounded sets $A, B \subseteq X$ such that A is compact and B is not. Avoid trivial choices for A; for example, A should not be a subset of a finite-dimensional subspace.

   (10 points)

   Solution. Let $||\cdot||$ denote the norm on X. As proven in class, the unit ball $B = \{x \in X : ||x|| \leq 1\}$ is non-compact, for *any* infinite-dimensional X. On the other hand, if $\{\mathbf{a}_n : n \geq 1\}$ is a convergent sequence in X, with limit $\mathbf{a}$, then $A = \{\mathbf{a}_n : n \geq 1\} \cup \{\mathbf{a}\}$ is a set in which every sequence has a convergent subsequence. Hence A is compact. In order to ensure a nontrivial choice of A, we could pick $X = \ell^2$, and

   $$\mathbf{a}_n = \left(1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots\right), \qquad \mathbf{a} = \left(1, \frac{1}{2}, \frac{1}{3}, \cdots\right),$$

   so that A does not lie in any finite-dimensional subspace. $\square$