Math 421/510 Quiz 3 Solution

Name: **SID #:**

1. Recall that a linear functional ℓ on a normed vector space X is *bounded* if there exists a finite constant $C > 0$ such that
$$|\ell(x)| \leq C||x||_X, \qquad \text{for all } x \in X.$$
For a normed vector space X of your choice, find a linear functional on X that is bounded and one that is not. Provide adequate reasoning for your answer.

(10 points)

Solution. Let X denote the real vector space of all polynomials in $[0, 1]$, equipped with the sup norm. Let us define two linear functionals ℓ_1 and ℓ_2 on X, with
$$\ell_1(p) = p(1) \quad \text{and} \quad \ell_2(p) = p'(1).$$
The functional ℓ_1 is bounded, since
$$|\ell_1(p)| = |p(1)| \leq \sup_{t\in[0,1]} |p'(t)| = ||p||_\infty.$$
However, ℓ_2 is not. To see this, consider the sequence $p_n \in X$ given by $p_n(t) = t^n$. We observe that
$$||p_n||_\infty = 1 \text{ for all } n \geq 1, \quad \text{whereas} \quad \ell_2(p_n) = p_n'(1) = n.$$
Since
$$\frac{|\ell_2(p_n)|}{||p_n||_\infty} = n \to \infty,$$
ℓ_2 is unbounded. □