Math 421/510 Quiz 7 Solution

1. Consider the linear space X consisting of all real sequences $\mathbf{x} = \{x_n\}$ that are absolutely summable. Equip X with the two norms $||\cdot||_1$ and $||\cdot||_2$:

$$||\mathbf{x}||_1 = \sum_n |x_n|, \qquad ||\mathbf{x}||_2 = \Big(\sum_n |x_n|^2\Big)^{\frac{1}{2}}.$$

Show that the identity map from $(X, ||\cdot||_1)$ onto $(X, ||\cdot||_2)$ is continuous but not open. Why does this not contradict the open mapping theorem?

Solution. For any $N \geq 1$, and any choice of scalars $\{x_1, \cdots, x_N\}$, we have the easy inequality

$$\sum_{n \leq N} |x_n|^2 \leq \Big(\sum_{n \leq N} |x_n|\Big)^2 \qquad \text{or} \qquad \Big(\sum_{n \leq N} |x_n|^2\Big)^{\frac{1}{2}} \leq \sum_{n \leq N} |x_n|.$$

Letting $N \to \infty$, this gives

$$||x||_2 \leq ||x||_1$$

for every $\mathbf{x} \in X$. In other words, the identity map is continuous from $(X, ||\cdot||_1)$ to $(X, ||\cdot||_2)$.

Suppose if possible that the above map is also open. That would imply that the identity map $(X, ||\cdot||_2)$ to $(X, ||\cdot||_1)$ is continuous, i.e. there exists a constant $C > 0$ such that

$$||\mathbf{x}||_1 \leq C||\mathbf{x}||_2 \quad \text{for all } \mathbf{x} \in X. \tag{1}$$

However, such an inequality is clearly false, as can be seen by choosing

$$\mathbf{x}_N = \Big(1, \frac{1}{2}, \cdots, \frac{1}{N}, 0, 0 \cdots\Big),$$

the successive terms of a truncated harmonic series. The left hand side of (1) grows like $\log N$ whereas the right hand side is bounded by a fixed constant independent of N. Letting $N \to \infty$, we reach a contradiction, which establishes that the map is not open.

The normed space $(X, ||\cdot||_2)$ is not complete. So there is no contradiction with the open mapping theorem, which requires both domain and range spaces to be Banach. $\square$