Math 421/510 Quiz 8 Solution

1. Let c_0 denote the linear space, equipped with sup norm, of all real sequences that tend to zero. Describe the spaces c_0^* and c_0^{**}, and use these to determine whether c_0 is reflexive.

Solution. We will show that c_0^* is isometrically isomorphic to $\ell^1(\mathbb{N})$, the space of all absolutely summable real sequences. By a theorem stated in class, we then have that $c_0^{**} = (\ell^1(\mathbb{N}))^* \cong \ell^\infty(\mathbb{N})$. Since the latter space is strictly larger than c_0, this shows that c_0 is not reflexive.

Let $\Phi : \ell^1(\mathbb{N}) \to c_0^*$ be given by

$$\Phi(\boldsymbol{\alpha}) = T_{\boldsymbol{\alpha}}, \quad \text{where} \quad T_{\boldsymbol{\alpha}}(\mathbf{x}) = \sum_n \alpha_n x_n, \quad \mathbf{x} = (x_n) \in c_0, \ \boldsymbol{\alpha} = (\alpha_n) \in \ell^1.$$

Clearly, $T_{\boldsymbol{\alpha}}$ is linear, with $|T_{\boldsymbol{\alpha}}(\mathbf{x})| \leq ||\mathbf{x}||_\infty ||\boldsymbol{\alpha}||_1$, so $||T_{\boldsymbol{\alpha}}||_{\mathrm{op}} \leq ||\boldsymbol{\alpha}||_1$. Choosing sequences $\mathbf{x}^{(N)}$ whose entries are of the form

$$x_n^{(N)} = \begin{cases} 1 & \text{if } n \leq N \text{ and } \alpha_n > 0, \\ -1 & \text{if } n \leq N \text{ and } \alpha_n < 0, \\ 0 & \text{otherwise,} \end{cases}$$

we see that $||\mathbf{x}^{(N)}||_\infty = 1$ for all sufficiently large N and that

$$T_{\boldsymbol{\alpha}}(\mathbf{x}^{(N)}) = \sum_{n=1}^N |\alpha_n| \longrightarrow ||\boldsymbol{\alpha}||_1 \quad \text{as } N \to \infty.$$

This shows that $||T_{\boldsymbol{\alpha}}|| \geq ||\boldsymbol{\alpha}||_1$, and hence $||T_{\boldsymbol{\alpha}}||_{\mathrm{op}} = ||\boldsymbol{\alpha}||_1$.

It remains to show that Φ is surjective, i.e., any $T \in c_0^*$ is of the form $T = T_{\boldsymbol{\alpha}}$ for some $\boldsymbol{\alpha} \in \ell^1(\mathbb{N})$. Define $\alpha_n = T(\mathbf{e}_n)$, where $\mathbf{e}_n$ is the n-th canonical basis vector. Let $\mathcal{E}$ denote the span of the vectors $\{\mathbf{e}_n\}$, i.e., the vector space of all sequences that have only finitely many nonzero terms. We observe that for each $x \in \mathcal{E}$,

$$T(\mathbf{x}) = T(\sum_n x_n \mathbf{e}_n) = \sum_n x_n T(\mathbf{e}_n) = \sum_n x_n \alpha_n = T_{\boldsymbol{\alpha}}(\mathbf{x}).$$

Choosing sequences $\mathbf{x}^{(N)}$ as above and recalling that T is bounded, we see that for all sufficiently large $N \geq 1$,

$$\sum_{n=1}^N |\alpha_n| = T_{\boldsymbol{\alpha}}(\mathbf{x}^{(N)}) = T(\mathbf{x}^{(N)}) \leq ||T|| < \infty, \quad \text{hence } \boldsymbol{\alpha} \in \ell^1(\mathbb{N}).$$

Since the two bounded linear operators T and $T_{\boldsymbol{\alpha}}$ agree on the dense subspace $\mathcal{E}$ of c_0, we conclude that $T = T_{\boldsymbol{\alpha}}$ as claimed. $\square$