Final Exam Practice Problems

---

1. Prove that the series
$$\sum_{n=0}^{\infty}\left[\frac{x^{2n+1}}{2n+1}-\frac{x^{n+1}}{2n+2}\right]$$
converges pointwise but not uniformly on $[0,1]$.

2. Prove that the series
$$\sum_{n=1}^{\infty}\frac{x}{n^{\alpha}(1+nx^2)}$$
converges uniformly on every finite interval in $\mathbb{R}$ if $\alpha>\frac{1}{2}$. Is the convergence uniform on $\mathbb{R}$?

3. Define two sequences f_n and g_n as follows:
$$f_n(x)=x\left(1+\frac{1}{n}\right)\quad\text{if } x\in\mathbb{R},\ n=1,2,\cdots$$
$$g_n(x)=\begin{Bmatrix}\frac{1}{n} & \text{if } x=0 \text{ or if } x \text{ is irrational,}\\ b+\frac{1}{n} & \text{if } x \text{ is rational with } x=\frac{a}{b},\end{Bmatrix}$$
where, in the last line above, a,b are integers that are relatively prime, and $b>0$. Set $h_n(x)=f_n(x)g_n(x)$.
   (a) Prove that f_n and g_n converge uniformly on every bounded interval.
   (b) Prove that h_n does not converge uniformly on any bounded interval.

4. Define the Fourier transform as follows:
$$\widehat{f}(\xi)=\int_{\mathbb{R}}f(x)e^{-2\pi ix\xi}\,dx.$$
   (a) Assuming that all integrals are absolutely convergent, show that the inverse Fourier transform is given by the formula:
$$f(x)=\int e^{2\pi ix\xi}\widehat{f}(\xi)\,d\xi.$$
   (b) Let f be a function that is both absolutely integrable and square integrable. State and prove a version of Plancherel's theorem connecting $||f||_2$ and $||\widehat{f}||_2$. Here $||\cdot||_2$ is given by
$$||g||_2=\int_{\mathbb{R}}|g(x)|^2\,dx.$$
   (c) Prove the Riemann-Lebesgue lemma: given a function f that is absolutely integrable on $\mathbb{R}$, show that $\widehat{f}(\xi)\to 0$ as $|\xi|\to\infty$.

(d) In the second midterm, you used Plancherel's theorem for a 2π-periodic Riemann integrable function f to show that the Fourier coefficients of f tend to zero. Explore whether a similar proof would work here. In other words, can (c) be deduced from (b)?

5. Given a function $f : \mathbb{Z}_N = \{0, 1, \cdots, N-1\}$, define its discrete Fourier transform as follows:

$$\widehat{f}(n) = \sum_{k=0}^{N-1} f(k) e^{-\frac{2\pi i k}{N}}.$$

(a) Find a formula for the inverse of the discrete Fourier transform that expresses f in terms of $\widehat{f}$.

(b) State and prove analogues of Plancherel's theorem and Parseval's theorem for f and $\widehat{f}$.

6. Assume that $f \in \mathcal{R}[a, b]$. Prove that

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \text{ exists and has the value } \int_a^b f(x)\,dx.$$

Deduce that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{\pi}{4}, \qquad \lim_{n\to\infty} \sum_{k=1}^{n} (n^2+k^2)^{-\frac{1}{2}} = \log(1+\sqrt{2}).$$

7. Let p_n be a polynomial of degree m_n and suppose that p_n converges uniformly to f on the compact interval $[a, b]$, where f is not a polynomial. Show that $m_n \to \infty$.

8. Suppose that $f : [1, \infty) \to \mathbb{C}$ is continuous and that $\lim_{x\to\infty} f(x)$ exists. True or false: there exists a sequence of polynomials p_n such that

$$p_n(1/x) \longrightarrow f(x) \text{ uniformly on } [1, \infty).$$

9. Does there exist a sequence of polynomials p_n such that $p_n \to 0$ pointwise on $[0, 1]$, but

$$\int_0^1 p_n(x)\,dx \to 3?$$

10. Fix $\alpha \in (0, 1]$. Given a constant $K > 0$, let us recall that $f \in \mathrm{Lip}_\alpha([0, 1]; K)$ if

$$|f(x) - f(y)| \le K|x-y|^\alpha \text{ for all } x, y \in [0, 1].$$

Let us denote by Lip_α the class of all functions on $[0, 1]$ that belong to $\mathrm{Lip}_\alpha([0, 1]; K)$ for some K.

(a) Is Lip_α a subspace of $C[0, 1]$? Is it a subalgebra?

(b) Show that Lip_α is not closed in $C[0, 1]$.

(c) Show that Lip_α is, on one hand, dense in $C[0,1]$, and also of first category (i.e. a countable union of nowhere dense sets) in $C[0,1]$.

(d) Find a norm on Lip_α under which the space is complete.

11. For K and α fixed, show that

$$\{f \in \mathrm{Lip}_\alpha([0,1];K) : f(0) = 0\}$$

is a compact subset of $C[0,1]$.

12. Let f be a positive continuous function on the compact interval $[a,b]$. Determine whether the following limit exists; if it does, find the limit

$$\lim_{n\to\infty} \left[\int_a^b f(x)^n \, dx\right]^{\frac{1}{n}}.$$

13. Suppose that β_n is a bounded sequence in $\mathrm{BV}[a,b]$, with $||\beta_n||_{\mathrm{BV}} \leq K$. Show that some subsequence (α_n) of (β_n) converges pointwise to a function $\alpha \in \mathrm{BV}[a,b]$ with $||\alpha||_{\mathrm{BV}} \leq K$, and that

$$\int_a^b f d\alpha_n \longrightarrow \int_a^b f \, d\alpha \quad \text{for all } f \in C[a,b].$$

14. Given a sequence (x_n) of distinct points in (a,b) and a sequence (c_n) of real numbers with $\sum_n |c_n| < \infty$, define α by

$$\alpha(x) = \sum_n c_n I(x - x_n), \qquad \text{where } I(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Show that $f \in \mathcal{R}_\alpha[a,b]$ for every $f \in C[a,b]$; then evaluate

$$\int_a^b f \, d\alpha$$

in terms of c_n and $f(x_n)$.

15. Determine whether the following statement is true or false: Let A be an open subset of $\mathbb{R}^n$. Suppose that $\mathbf{f} : A \to \mathbb{R}^n$ is a continuously differentiable function on A that has nonvanishing Jacobian at every point in A. Then $\mathbf{f}$ is an open map, i.e., carries open sets to open sets. Recall that the Jacobian of $\mathbf{f}$ is the determinant of the first derivative $\mathbf{f}'$ of $\mathbf{f}$.

16. Let α be non-decreasing and let $f \in \mathcal{R}_\alpha[a,b]$. Define

$$F(x) = \int_a^x f(x) d\alpha(x).$$

Prove the following version of the fundamental theorem of calculus, adapted to Riemann-Stieltjes integrals:

(a) $F \in \mathrm{BV}[a,b]$.

(b) F is continuous at each point where α is continuous.

(c) F is differentiable at each point where α is differentiable and f is continuous. At any such point $F'(x) = f(x)\alpha'(x)$.

17. Determine whether the following statement is true or false. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin\left(1 + \frac{x}{n}\right)$$

converges uniformly on $\mathbb{R}$.

18. (a) Show that the Fejer kernel K_n can be written as

$$K_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) e^{ikx}.$$

(b) Let $\sigma_n(f) = K_n * f$. Show that for any continuous, 2π-periodic f,

$$||\sigma_n(f)||_2 \leq ||f||_2 \quad \text{and } ||\sigma_n(f)||_\infty \leq ||f||_\infty.$$

(c) If $f \in \mathcal{R}[-\pi, \pi]$, show that $\sigma_n(f)(x) \to f(x)$ for every point of continuity x of f.