Math 321 Assignment 10
Due Wednesday, March 27 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Define $f(0,0) = 0$, and
$$f(x,y) = \frac{x^3}{x^2+y^2} \qquad \text{if } (x,y) \neq (0,0).$$
   (a) For any unit vector $\mathbf{u} \in \mathbb{R}^2$, prove that the directional derivative $D_{\mathbf{u}} f(0,0)$ given by
$$D_{\mathbf{u}} f(x_0,y_0) = \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f((x_0,y_0)+t\mathbf{u}) - f(x_0,y_0)}{t}$$
   exists, and is bounded by 1 in absolute value. In particular, all the partial derivatives of f exist and are bounded.

   (b) In spite of this, show that f is not differentiable at $(0,0)$.

2. The continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$. Here is an example. If
$$f(t) = \begin{cases} t + 2t^2 \sin\left(\frac{1}{t}\right) & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$
and $f'(0) = 1$, then show that f' is bounded in $(-1,1)$ but f is not one-to-one in any neighbourhood of 0.

3. Use the inverse function theorem to show that each of the following functions f is locally invertible at every point of its domain D. Then show that it is in fact globally invertible by computing f^{-1} explicitly.

   (a) $f : D = \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = -1/\bar{z}$.

   (b) $\mathbf{f} = (f_1, f_2, f_3)$ on $D = \mathbb{R}^3 \setminus \{\mathbf{x} = (x_1,x_2,x_3) : x_1+x_2+x_3 = -1\}$:
$$f_k(x_1,x_2,x_3) = \frac{x_k}{1+x_1+x_2+x_3}, \qquad k = 1,2,3.$$

4. (a) State conditions on f and g which will ensure that the equations
$$x = f(u,v), \qquad y = g(u,v)$$
   can be solved for u and v in a neighbourhood of (x_0,y_0). If the solutions are $u = F(x,y)$, $v = G(x,y)$ and if $J = \partial(f,g)/\partial(u,v) = \det((f,g)')$, show that
$$\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v}, \qquad \frac{\partial F}{\partial y} = -\frac{1}{J}\frac{\partial f}{\partial v}, \qquad \frac{\partial G}{\partial x} = -\frac{1}{J}\frac{\partial g}{\partial u}, \qquad \frac{\partial G}{\partial y} = \frac{1}{J}\frac{\partial f}{\partial u}.$$

(b) Compute J and the partial derivatives of F and G at $(x_0, y_0) = (1, 1)$ when $f(u, v) = u^2 - v^2$, $g(u, v) = 2uv$.

5. Let $f = u + iv$ be a complex-valued function satisfying the following conditions: $u \in C^1$ and $v \in C^1$ on the open disk $A = \{z \in \mathbb{C} : |z| < 1\}$; f is continuous on the closed disk $\overline{A} = \{z \in \mathbb{C} : |z| \leq 1\}$, and

$$u(x, y) = x, \quad v(x, y) = y \quad \text{whenever} \quad x^2 + y^2 = 1,$$

the Jacobian $J_f(z) =$ determinant of $f'(z) > 0$ for all $z \in A$. Let $B = f(A)$ denote the image of f under A. Prove that

(a) f is an open map, i.e., if X is an open subset of A, then $f(X)$ is an open subset of B.

(b) B is an open disk of radius 1.

(c) For each point $u_0 + iv_0$ in B, there is only a finite number of points $z \in A$ such that $f(z) = u_0 + iv_0$.