Math 321 Assignment 4
Due Wednesday, January 30 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Let $C^1([a,b];\mathbb{C})$ denote the vector space of complex-valued, differentiable functions on $[a,b]$ whose first derivative is continuous. There is a natural metric on $C^1([a,b];\mathbb{C})$, namely

$$d(f,g) = ||f-g||_\infty + ||f'-g'||_\infty, \quad \text{where} \quad ||f||_\infty = \sup_{x\in[a,b]} |f(x)|.$$

You do not need to prove that d is a metric. Determine whether polynomials are dense in $C^1([a,b];\mathbb{C})$, with respect to the topology mentioned above. Can you generalize this to $C^k([a,b];\mathbb{C})$ for integers $k \geq 1$?

2. For each of the statements below, determine whether it is true or false, with proper justification.

(a) Let $\{f_n : n \geq 1\}$ be a sequence of differentiable functions (either real or complex-valued) on $[a,b]$ whose first derivatives are uniformly bounded on $[a,b]$. Then some subsequence of $\{f_n : n \geq 1\}$ must be uniformly convergent on $[a,b]$.

(b) Suppose that $\{f_n : n \geq 1\}$ is an equicontinuous sequence of functions (either real or complex-valued) on $[a,b]$ with continuous first derivatives. Then the sequence of derivatives $\{f'_n : n \geq 1\}$ must be uniformly bounded.

(c) If $\mathcal{F} \subseteq C([a,b];\mathbb{C})$ is equicontinuous, then its closure $\overline{\mathcal{F}}$ is also equicontinuous.

(d) Given an integer $n \geq 0$ and a function $f \in C([0,1];\mathbb{C})$, its n-th moment is defined to be

$$M_n(f) = \int_0^1 x^n f(x)\,dx.$$

There exists a function $f \in C([0,1];\mathbb{C})$, $f \not\equiv 0$ with all vanishing moments, i.e., $M_n(f) = 0$ for all $n \geq 0$.

(e) Define $T : C([a,b];\mathbb{C}) \to C([a,b];\mathbb{C})$ by

$$Tf(x) = \int_a^x f(t)\,dt.$$

Then T maps bounded sets in $C([a,b];\mathbb{C})$ into equicontinuous sets in $C([a,b];\mathbb{C})$.

3. Let $\mathcal{C}^{2\pi}(\mathbb{C})$ denote the class of continuous, complex-valued, 2π-periodic functions on $\mathbb{R}$, equipped with the metric

$$d(f,g) = \sup_{x\in[0,2\pi]} |f(x) - g(x)|.$$

Use Weierstrass's first approximation theorem (namely, polynomials are dense in $C([a,b];\mathbb{C})$) to prove his second one, namely, *trigonometric polynomials are dense in* $\mathcal{C}^{2\pi}(\mathbb{C})$. Recall that a trigonometric polynomial is a function of the form

$$T(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\,,$$

where the coefficients $a_0, a_1, b_1, \cdots, a_n, b_n$ are complex numbers. *Hint:* First try approximating an even function $f \in \mathcal{C}^{2\pi}(\mathbb{C})$.

4. (a) Set $\mathbb{T} = \{e^{i\theta} : \theta \in \mathbb{R}\} = \{z \in \mathbb{C} : |z| = 1\}$. Show that the space of polynomials in $z = e^{i\theta}$ with complex coefficients separates points in $\mathbb{T}$ and vanishes at no point of $\mathbb{T}$, but is not dense in $C(\mathbb{T};\mathbb{C})$ = the space of continuous, complex-valued functions on $\mathbb{T}$. Explain why this does not contradict the Stone-Weierstrass theorem.

   (b) Use the Stone-Weierstrass theorem proved in class to formulate and prove a version of the same theorem that will help us understand the example in part (a). More precisely, given a compact metric space (X,d), find a set of sufficient conditions for an algebra $\mathcal{A} \subseteq C(X;\mathbb{C})$ to be dense in $C(X;\mathbb{C})$. Are these conditions also necessary? Here $C(X;\mathbb{C})$ denotes the space of complex-valued, continuous functions on X, equipped with the supremum norm.

   (c) Use the result you proved in part (b) to give a second proof of Problem 3.