Math 321 Assignment 5
Due Wednesday, February 13 at 9AM on Canvas

Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Given a nonconstant, non-decreasing function $\alpha : [a,b] \to \mathbb{R}$, let $\mathcal{R}_\alpha[a,b]$ denote the collection of all bounded, real-valued functions on $[a,b]$ that are Riemann-Stieltjes integrable with respect to α. Determine whether $\mathcal{R}_\alpha[a,b]$ is

   (a) a vector space,

   (b) a lattice (this means if $f \in \mathcal{R}_\alpha[a,b]$, then $|f| \in \mathcal{R}_\alpha[a,b]$),

   (c) an algebra.

2. This problem focuses on computing the Riemann-Stieltjes integral for specific integrators.

   (a) Let $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ be a fixed collection of points in $[a,b]$, and let α be a non-decreasing step function on $[a,b]$ that is constant on each of the open intervals (x_{i-1}, x_i) and has jumps of size $\alpha_i = \alpha(x_i+) - \alpha(x_i-)$ at each of the points x_i. For $i = 0$ and n, we make the obvious adjustments

   $$\alpha_0 = \alpha(a+) - \alpha(a), \qquad \alpha_n = \alpha(b) - \alpha(b-).$$

   If a bounded function $f : [a,b] \to \mathbb{R}$ is continuous at each of the points x_i, show that $f \in \mathcal{R}_\alpha[a,b]$ and

   $$\int_a^b f\, d\alpha = \sum_{i=0}^{n} f(x_i)\alpha_i.$$

   (b) If a function f is continuous on $[1,n]$, compute $\int_1^n f(x)d[x]$, where $[x]$ is the greatest integer in x. What is the value of $\int_1^t f(x)d[x]$ if t is not an integer?

   (c) Suppose that $f \in C([a,b];\mathbb{R})$ and α is continuously differentiable on $[a,b]$ with a non-negative first derivative. Express the Riemann-Stieltjes integral

   $$\int_a^b f d\alpha$$

   as a Riemann integral involving f and α'.

3. Explain whether each of the following statements is true or false.

   (a) The function $\chi_{\mathbb{Q}}$ is Riemann integrable on $[0,1]$. *Remark: Given any set E, the notation χ_E denotes the characteristic or indicator function on E: more precisely,*

   $$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The function χ_Δ is Riemann integrable on $[0,1]$, where Δ is the Cantor middle-third set.

(c) $\bigcap_\alpha \{\mathcal{R}_\alpha[a,b] : \alpha \text{ a non-decreasing integrator}\} = C([a,b];\mathbb{R})$.

(d) If f is a monotone function and α is both continuous and non-decreasing, then $f \in \mathcal{R}_\alpha[a,b]$.

(e) There exists a non-decreasing function $\alpha : [a,b] \to \mathbb{R}$ and a function $f \in \mathcal{R}_\alpha[a,b]$ such that f and α share a common-sided point of discontinuity. Recall that f and α are said to share a left-sided (respectively right-sided) discontinuity at a point $x_0 \in [a,b]$ if both f and α are discontinuous from the left (respectively right) at x_0.

(f) Let $f \in \mathcal{R}_\alpha[a,b]$, and m, M be two constants such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. If φ is continuous on $[m,M]$, then $\varphi \circ f \in \mathcal{R}_\alpha[a,b]$.

(g) Let $\mathrm{BV}[a,b]$ denote the space of real-valued functions on $[a,b]$ that are of bounded variation. Then $\mathrm{BV}[a,b]$ is complete under the norm $||\cdot||_{\mathrm{BV}}$:

$$||f||_{\mathrm{BV}} = |f(a)| + V_a^b f, \qquad \text{where } V_a^b f \text{ denotes the total variation of } f.$$