**Math 321 Assignment 8**
**Due Wednesday, March 6 at 9AM on Canvas**

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Instructions

(i) Solutions should be well-crafted, legible and written in complete English sentences. You will be graded both on accuracy as well as the quality of exposition.

(ii) Theorems stated in the text and proved in class do not need to be reproved. Any other statement should be justified rigorously.

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1. Given a right continuous integrator $\alpha \in \mathrm{BV}[a,b]$, let $L_\alpha : C[a,b] \to \mathbb{R}$ be the linear functional given by

$$L_\alpha(f) = \int_a^b f \, d\alpha.$$

   (a) Show that $L_\alpha$ is continuous. (*Remark: This is the easy direction of the Riesz representation theorem, which we left as an exercise in class.*)

   (b) Show that $||L_\alpha|| = V_a^b \alpha$. Here $||L_\alpha||$ denotes the operator norm of $L_\alpha$, given by

$$||L_\alpha|| = \sup \left\{ |L_\alpha(f)| : f \in C[a,b], \; ||f||_\infty = \sup_{x \in [a,b]} |f(x)| = 1 \right\}.$$

   *Remark: The Riesz representation theorem provides, via the mapping $L = L_\alpha \mapsto \alpha$, a bijection between $C[a,b]^*$ = the dual of $C[a,b]$ and the space of right continuous integrators $\alpha \in BV[a,b]$ with $\alpha(a) = 0$. The result above shows that this mapping preserves lengths, i.e., is an isometric isomorphism!*

2. Let $f$ be a twice differentiable $2\pi$-periodic function with continuous first and second derivatives. Is $f$ the uniform limit of its partial Fourier sums?

3. Determine whether the following statement is true or false: If $f : \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic and Riemann-integrable on $[-\pi, \pi]$, then $||f_\epsilon - f||_2 \to 0$ as $\epsilon \to 0$. Here $f_\epsilon$ denotes the translated function $f_\epsilon(x) = f(x + \epsilon)$.

4. (a) Obtain the Fourier series of the $2\pi$-periodic function $g$ that coincides with $f(t) = (\pi - t)^2$ on $[0, 2\pi]$.

   (b) Does $g$ match its Fourier series? Give reasons for your answer.

   (c) Use your results from above to derive the identity: $\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$.

5. (a) Show that the Fourier series of a function $f$ can alternatively be written in the form

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikx}, \qquad \text{where} \qquad \widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt$$

   is referred to as the $k$th Fourier coefficient.

   (b) Determine the relation of $\widehat{f}(k)$ with $\widehat{g}(k)$ in each of the following cases:

(i) $g$ is a translate of $f$, namely $g(x) = f(x + \alpha)$.

(ii) $g$ is a modulation of $f$, namely $g(x) = f(x)e^{-i\alpha x}$, where $\alpha \in \mathbb{Z}$.

(c) Given two bounded, $2\pi$-periodic functions $f$ and $g$, both of which are Riemann-integrable on $[-\pi, \pi]$, define their convolution $h = f * g$ as follows,

$$h(x) = f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t)g(t)\, dt.$$

Find $\widehat{h}$ in terms of $\widehat{f}$ and $\widehat{g}$.

(d) Note that the $n$th partial Fourier sum $s_n f$ and the $n$th Cesàro sum $\sigma_n f$ are both given in terms of convolutions of $f$ with appropriate kernels, i.e.,

$$s_n f = f * D_n, \qquad \sigma_n f = f * F_n.$$

Recall that $\sigma_n f = (s_1 f + s_2 f + \cdots + s_n f)/n$. Derive explicit formulae for the convolution kernels $D_n$ and $F_n$, known respectively as Dirichlet and Fejér kernels.