Midterm 1 Solutions

1. (a) **Let (X, d) be a metric space, and let $\mathcal{C}(X;\mathbb{C})$ denote the space of continuous, complex-valued functions on X. When is a family of functions $\mathcal{F} \subseteq \mathcal{C}(X;\mathbb{C})$ said to be *equicontinuous at a point* $x_0 \in X$?**

   Solution. We say that a family of functions $\mathcal{F} \subseteq \mathcal{C}(X;\mathbb{C})$ is *equicontinuous at a point* $x_0 \in X$ if for every $\epsilon > 0$, there exists a positive number δ, depending only on x_0 and ϵ, such that for all $f \in \mathcal{F}$,

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad d(x, x_0) < \delta. \tag{1}$$

□

   (b) **Give an example, with justification, of an infinite family of non-constant functions that is equicontinuous at a point.**

   Solution. Suppose that $X = [0, 1]$, equipped with the standard Euclidean metric. Then the class of functions $\mathcal{F} = \{f_n : n \geq 1\}$ given by

$$f_n(x) = \frac{x^n}{n}, \quad x \in [0, 1]$$

   We claim that $\mathcal{F}$ is equicontinuous at every point $x_0 \in [0, 1]$. To see this, fix any $\epsilon > 0$. Then for all $n > 2/\epsilon$, we have

$$|f_n(x) - f_n(x_0)| \leq |f_n(x)| + |f_n(x_0)| \leq \frac{2}{n}.$$

   On the other hand, each f_n, being a monomial, is continuous at x_0; so there exists $\delta_n > 0$ such that

$$|f_n(x) - f_n(x_0)| < \epsilon \quad \text{whenever } |x - x_0| < \delta_n.$$

   Choosing $\delta = \min\{\delta_n : 1 \leq n \leq 2/\epsilon\}$ ensures that the requirement (1) is met for all $f \in \mathcal{F}$. □

   (c) **State the Arzelà-Ascoli theorem with all accompanying hypotheses. Define any terminology you need to use to state this theorem.**

   Solution. Let (X, d) be a compact metric space. Let us note that in this case $C(X;\mathbb{C})$ is a metric space endowed with the sup norm.

   The Arzela-Ascoli theorem asserts that any set $\mathcal{F} \subseteq C(X;\mathbb{C})$ is compact if and only if it is closed, uniformly bounded and equicontinuous.

   Recall that $\mathcal{F}$ is said to be *uniformly bounded* if $\sup\{|f(x)| : x \in X, f \in \mathcal{F}\} < \infty$. Equicontinuity has been defined in part (a) of this problem. □

   (d) **Give an example of a metric space X, and a subalgebra of $\mathcal{C}(X;\mathbb{R})$ that fails to separate points and also vanishes at some point.**

   Solution. Let $X = [0, 2\pi]$. The subalgebra $\mathcal{A}$ of $\mathcal{C}(X;\mathbb{R})$ generated by $\sin x$ is given by

$$\mathcal{A} = \Big\{f : f(x) = \sum_{j=1}^{n} a_j (\sin x)^j \text{ for some choice of } a_1, \cdots, a_n \in \mathbb{R} \text{ and } n \geq 1\Big\}.$$

   This algebra

- fails to separate points because $f(0) = f(2\pi)$ for all $f \in \mathcal{A}$;
- vanishes at 0, since $f(0) = 0$ for all $f \in \mathcal{A}$.

□

2. **Give brief answers to the following questions. The answer should be in the form of a short proof or an example, as appropriate.**

(a) **Determine whether the following statement is true or false: Every continuous function f in $\mathcal{C}[1,2]$ can be uniformly approximated by a sequence of even polynomials.**

Solution. The statement is true.

The class of even polynomials restricted to $[1,2]$ is a subalgebra of $C[1,2]$ that separates points (eg $f(x) = x^2$) and vanishes at no point (same f). Thus by Stone-Weierstrass theorem, it is dense in $C[1,2]$, i.e., every continuous function on $[1,2]$ can be uniformly approximated on this interval by a sequence of even polynomials. □

(b) **Determine whether the following statement is true or false: Every continuous function f in $\mathcal{C}[1,2]$ can be uniformly approximated by a sequence of odd polynomials.**

Solution. The statement is true.

Given any $f \in C[1,2]$, set $g(t) = f(t)/t$. Then $g \in C[1,2]$. By part (a), there exists an even polynomial P of the form

$$P(t) = \sum_{j=0}^{n} a_j x^{2j} \quad \text{such that} \quad |g(t) - P(t)| < \epsilon/2 \quad \text{for all } t \in [1,2].$$

Multiplying both sides of the inequality by t, we find that

$$|f(t) - tP(t)| < t\frac{\epsilon}{2} < \epsilon \quad \text{for all } t \in [1,2].$$

Since $tP(t)$ is an odd polynomial, we are done. □

(c) **Would your answers to parts (a) and (b) change if f lies in $\mathcal{C}[0,1]$? State your answers clearly and prove them.**

Solution. The answer to part (a) would not change since the class of even polynomials on $[0,1]$ remains an algebra that separates points and vanishes at no point. The answer to (b) would change since any odd polynomial must vanish at the origin, so it would not be possible to approximate a continuous function that is nonvanishing at the origin by a sequence of odd polynomials. □

(d) **Let $\{f_n : n \geq 1\}$ be a sequence in $\mathcal{C}([a,b];\mathbb{R})$ with no uniformly convergent subsequence. Define a function F_n as**

$$F_n(x) = \int_a^x \sin(f_n(t))\,dt, \qquad x \in [a,b].$$

Does $\{F_n : n \geq 1\}$ have a uniformly convergent subsequence?

Proof. Yes. We observe that the family of functions $\mathcal{F} = \{F_n : n \geq 1\}$ is

- uniformly bounded, since $||F_n||_\infty \leq (b-a)$;
- equicontinuous, since $|F_n(x) - F_n(y)| \leq |x - y|$.

Thus by Arzela-Ascoli theorem, $\overline{\mathcal{F}}$ is compact; in other words every infinite sequence in $\mathcal{F}$ has a convergent subsequence. □

3. **Evaluate with justification**

$$\lim_{n\to\infty} \int_0^1 \frac{\pi n + \sin nx}{2n + \cos(n^2 x)}\, dx.$$

(20 points)

Solution. If we can prove that the integrand converges uniformly to $\pi/2$ on $[0,1]$, then we can interchange limit and integration to conclude that the limit is $\pi/2$. To prove uniform convergence, we observe that

$$\Big|\frac{\pi n + \sin nx}{2n + \cos(n^2 x)} - \frac{\pi}{2}\Big| = \Big|\frac{2\sin nx - \pi\cos(n^2 x)}{2n + \cos(n^2 x)}\Big| \le \frac{2+\pi}{2n-1}.$$

The right hand side is independent of x, and converges to 0 as $n \to \infty$, establishing the desired uniform convergence. □

4. **Let $f : \mathbb{R}^n \to \mathbb{R}$ denote the function**

$$f(x) = e^{-|x|^2}, \qquad x = (x_1, \cdots, x_n), \quad |x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Can there exist a sequence $\{p_k\}$ of polynomials in n variables that converges to f uniformly on every compact subset of $\mathbb{R}^n$?

Solution. Such a sequence of polynomials exists. By the Stone-Weierstrass theorem, polynomials in n variables are dense in $C[-k,k]^n$ for every $k \ge 1$. Since f is continuous on $\mathbb{R}^n$, we can therefore find a polynomial p_k such that

$$|f(x) - p_k(x)| < \frac{1}{k} \quad \text{for all } x \in R_k = [-k,k]^n. \tag{2}$$

We claim that $p_k \to f$ uniformly on every compact subset of $\mathbb{R}^n$. Indeed given any compact set $R \subseteq \mathbb{R}^n$, there exists k_0 such that $R \subseteq R_k$ for all $k \ge k_0$. The condition (2) then implies that for all $k \ge k_0$,

$$\sup_{x\in R} |f(x) - p_k(x)| \le \sup_{x \in R_k} |f(x) - p_k(x)| < \frac{1}{k} \to 0 \text{ as } k \to \infty,$$

proving the claim. □