Math 321, Spring 2019
Midterm 1, February 1

Name: SID:

Instructions

- The total time is 50 minutes.
- The total score is 80 points.
- Use the reverse side of each page if you need extra space.
- Show all your work. A correct answer without intermediate steps will receive no credit.
- Partial credit will be assigned to the clarity and presentation style of solutions, so please ensure that your answers are effectively communicated.

Problem	**Points**	**Score**
1	30	
2	30	
3	20	
4 (Extra credit)	20	
MAX	80	

1. (a) Let (X, d) be a metric space, and let $\mathcal{C}(X; \mathbb{C})$ denote the space of continuous, complex-valued functions on X. When is a family of functions $\mathcal{F} \subseteq \mathcal{C}(X; \mathbb{C})$ said to be *equicontinuous at a point* $x_0 \in X$?

(7 points)

(b) Give an example, with justification, of an infinite family of non-constant functions that is equicontinuous at a point.

(8 points)

(c) State the Arzelà-Ascoli theorem with all accompanying hypotheses. Define any terminology you need to use to state this theorem.

(8 points)

(d) Give an example of a metric space X, and a subalgebra of $\mathcal{C}(X;\mathbb{R})$ that fails to separate points and also vanishes at some point.

(7 points)

2. Give brief answers to the following questions. The answer should be in the form of a short proof or an example, as appropriate.

   (a) Determine whether the following statement is true or false: Every continuous function f in $\mathcal{C}[1,2]$ can be uniformly approximated by a sequence of even polynomials.

   (6 points)

   (b) Determine whether the following statement is true or false: Every continuous function f in $\mathcal{C}[1,2]$ can be uniformly approximated by a sequence of odd polynomials.

   (6 points)

(c) Would your answers to parts (a) and (b) change if f lies in $\mathcal{C}[0,1]$? State your answers clearly and prove them.

(8 points)

(d) Let $\{f_n : n \geq 1\}$ be a sequence in $\mathcal{C}([a,b];\mathbb{R})$ with no uniformly convergent subsequence. Define a function F_n as

$$F_n(x) = \int_a^x \sin(f_n(t))\,dt, \qquad x \in [a,b].$$

Does $\{F_n : n \geq 1\}$ have a uniformly convergent subsequence?

(10 points)

3. Evaluate with justification

$$\lim_{n\to\infty}\int_0^1 \frac{\pi n + \sin nx}{2n + \cos(n^2 x)}\,dx.$$

(20 points)

4. (Extra credit) Let $f : \mathbb{R}^n \to \mathbb{R}$ denote the function

$$f(x) = e^{-|x|^2}, \qquad x = (x_1, \cdots, x_n), \quad |x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Can there exist a sequence $\{p_k\}$ of polynomials in n variables that converges to f uniformly on every compact subset of $\mathbb{R}^n$?

(20 points)