

Midterm 1 Review: Practice Problems

1. We used the following lemma in the proof of the Stone-Weierstrass theorem. Prove it.

“Let \mathcal{A} be an algebra of real-valued functions on some set X and suppose that \mathcal{A} separates points and vanishes at no point of X . Then given any two points x, y in X , $x \neq y$ and $a, b \in \mathbb{R}$, we can find a function $f \in \mathcal{A}$ with $f(x) = a$ and $f(y) = b$.”

Keep in mind that X need not be compact and the functions in \mathcal{A} need not be continuous.

2. The proof of Stone-Weierstrass theorem also involved the following proposition. Prove it.

“Given any metric space X , let $\mathcal{A} \subseteq B(X; \mathbb{R})$ denote a subalgebra of the space of bounded real-valued functions on \mathbb{R} . Show that the closure $\overline{\mathcal{A}}$ of \mathcal{A} is both a subalgebra and a sublattice (i.e., if $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$.”

Verify also that the same statement holds with $B(X; \mathbb{R})$ replaced by $C_b(X; \mathbb{R})$, the space of bounded, continuous real-valued functions on X .

3. Given an arbitrary metric space X , verify whether the following statement is true or false: $C(X; \mathbb{R})$ always separates points and vanishes at no point.
4. Find a compact metric space (X, d) and algebras $\mathcal{A}, \mathcal{B} \subseteq C(X; \mathbb{R})$ such that
- (a) \mathcal{A} separates points but vanishes at some point.
 - (b) \mathcal{B} vanishes at no point but fails to separate points.
5. Evaluate with justification

$$\lim_{n \rightarrow \infty} \int_0^\pi \frac{n + \sin nx}{3n - \sin^2 nx} dx.$$

6. For each $n \in \mathbb{N}$, you are given a differentiable function $f_n : \mathbb{R} \rightarrow \mathbb{R}$ that satisfies

$$f_n(0) = 2019, \quad |f'_n(t)| \leq 321 + |t|^{201} \text{ for all } t \in \mathbb{R}.$$

Prove that there exists $f : \mathbb{R} \rightarrow \mathbb{R}$ and a subsequence f_{n_k} with the following property: for every compact subset K of \mathbb{R} , $f_{n_k} \rightarrow f$ uniformly on K . Clearly identify the principal theorems and methods that you apply.

7. The Arzela-Ascoli theorem can be rephrased as: “if $\{f_n : n \geq 1\}$ is a sequence of real-valued, equicontinuous and uniformly bounded functions on a compact metric space X , then $\{f_n\}$ has a uniformly convergent subsequence”. A website you found claims that the following is a proof of this theorem. Determine whether the proof is correct or incorrect.

Proof. Let $\mathcal{F} = \{f_n : n \geq 1\}$. Since \mathcal{F} is equicontinuous and X is compact, given $\epsilon > 0$, there is $\delta > 0$ such that

$$(1) \quad d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \text{ for all } n.$$

Compact also implies totally bounded, so X is totally bounded. Hence we can find finitely many points $\{x_k : 1 \leq k \leq K\} \subseteq X$ such that

$$X = \bigcup_{k=1}^K B(x_k; \delta).$$

Recall that \mathcal{F} is known to be uniformly bounded (say by the finite constant M), so each one of the K sequences $\{f_n(x_k) : n \geq 1\}$ is an infinite sequence in $[-M, M]$. By the compactness of $[-M, M]$, and we can find subsequence $n_1 < n_2 < \cdots < n_\ell < \cdots \rightarrow \infty$ such that

$$\lim_{\ell \rightarrow \infty} f_{n_\ell}(x_k) \text{ exists for all } 1 \leq k \leq K.$$

Since any convergent sequence is Cauchy, we can find N such that for all $n_\ell, n_{\ell'} \geq N$,

$$(2) \quad |f_{n_\ell}(x_k) - f_{n_{\ell'}}(x_k)| < \frac{\epsilon}{3} \quad \text{for all } n_\ell, n_{\ell'} \geq N \text{ and for all } 1 \leq k \leq K.$$

We claim that $\{f_{n_\ell} : \ell \geq 1\}$ is the desired uniformly convergent subsequence. This will follow if we can show that $\{f_{n_\ell} : \ell \geq 1\}$ is uniformly Cauchy. To see this, given $x \in X$, first identify the index k such that $d(x, x_k) < \delta$. Then for all $n_\ell, n_{\ell'} \geq N$,

$$\begin{aligned} |f_{n_\ell}(x) - f_{n_{\ell'}}(x)| &\leq |f_{n_\ell}(x) - f_{n_\ell}(x_k)| + |f_{n_\ell}(x_k) - f_{n_{\ell'}}(x_k)| + |f_{n_{\ell'}}(x_k) - f_{n_{\ell'}}(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

In the estimation above, we have used the equicontinuity criterion (1) to bound the first and the third term, and the pointwise Cauchy criterion (2) to bound the second. This allows us to conclude that $\|f_{n_\ell} - f_{n_{\ell'}}\|_\infty < \epsilon$ for all $n_\ell, n_{\ell'} \geq N$, and we are done. \square