Midterm 2 Review: Practice Problems

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1. Let $\mathcal{S}$ denote the set of functions in $\mathcal{C}[-\pi,\pi]$ of the form
$$f(x) = a\sin x + b\sin 2x$$
where a and b are arbitrary real numbers. Let $g(x) = x$ for $x \in [-\pi,\pi]$. Find $f \in \mathcal{S}$ for which $||g - f||_2$ is smallest.

(*Answer:* $f(x) = 2\sin x - \sin 2x$.)

2. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be the function
$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 2y & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Compute the lower and upper Riemann integrals
$$\underline{\int_0^1} f(x,y)\,dx \qquad \text{and} \qquad \overline{\int_0^1} f(x,y)\,dx$$
in terms of y.

(b) Show that
$$\int_0^1 f(x,y)\,dy \text{ exists for each fixed } x.$$
Compute
$$\int_0^t f(x,y)\,dy \text{ in terms of } (x,t) \in [0,1]\times[0,1].$$

(c) Define
$$F(x) = \int_0^1 f(x,y)\,dy.$$
Show that $\int_0^1 F(x)\,dx$ exists and find its value.

(d) There must be a moral to this long-winded story. What is it?

3. A certain Riemann-integrable function $f : [-\pi,\pi] \to \mathbb{C}$ and a complex sequence $\{c_k\}$ obey
$$\Big|\Big| f(t) - \sum_{k=-n}^{n} c_k e^{ikt} \Big|\Big|_2 \longrightarrow 0 \qquad \text{as} \qquad n \to \infty.$$
Prove the following statements:

(a) For any $g : [-\pi,\pi] \to \mathbb{C}$ with $g \in \mathcal{R}[-\pi,\pi]$,
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt = \sum_{k=-\infty}^{\infty} c_k\overline{\widehat{g}(k)}, \text{ where } \widehat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-ikt}\,dt.$$

(b) $c_k = \widehat{f}(k)$ and $\sum_k |c_k|^2 < \infty$.

4. Evaluate the following, with careful justification of all steps:

$$\sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} t^5 e^{-int}\, dt \right|^2$$

(*Answer:* $\frac{4\pi^{12}}{11}$.)

5. Let $g : [0,1] \to \mathbb{R}$ be bounded and $\alpha : [0,1] \to \mathbb{R}$ be nondecreasing. Assume that $g \in \mathcal{R}_\alpha[\delta, 1]$ for every $\delta > 0$.
   (a) Show that $g \in \mathcal{R}_\alpha[0,1]$ if α is continuous at 0.
   (b) Give an example of a pair (g, α) which shows that the conclusion of part (a) is false if α is not assumed to be continuous at 0.

6. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos nx + b_n \sin nx\big)$$

be the Fourier series of a function $f \in \mathrm{BV}[-\pi, \pi]$. Show that $\{na_n\}$ and $\{nb_n\}$ are bounded sequences.

7. Determine whether or not the following functions f are of bounded variation on $[0,1]$.
   (a) $f(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0$, $f(0) = 0$.

   (*Answer: Yes.*)

   (b) $f(x) = \sqrt{x} \sin(\frac{1}{x})$ if $x \neq 0$, $f(0) = 0$.

   (*Answer: No.*)

8. A function $f : [a,b] \to \mathbb{R}$ is said to satisfy a Lipschitz or Hölder condition of order $\alpha > 0$ if there exists $M > 0$ such that

$$|f(x) - f(y)| < M|x - y|^\alpha \text{ for all } x, y \in [a,b].$$

   (a) If f is such a function, show that $\alpha > 1$ implies that f is constant on $[a,b]$, whereas $\alpha = 1$ implies $f \in \mathrm{BV}[a,b]$.
   (b) Give an example of a function not of bounded variation satisfying a Hölder condition of order $\alpha < 1$.
   (c) Give an example of a function of bounded variation on $[a,b]$ that satisfies no Lipschitz condition on $[a,b]$.